Principles of Linear Algebra With *Maple*™
The Newton–Raphson Method

Kenneth Shiskowski and Karl Frinkle

Contents

Chapter 1

The Newton-Raphson Method for a Single Equation

1.1 The Geometry of the Newton-Raphson Method

In studying astronomy, Sir Isaac Newton needed to solve an equation $f(x) = 0$ involving trigonometric functions such as sine. He could not do it by any algebra he knew and all he really needed was just a very good approximation to the equation's solution. He then discovered the basic algorithm called the *Newton-Raphson Method* although Newton found it in a purely algebraic format which was very difficult to use and understand. The general method and its geometric basis was actually first seen by Joseph Raphson (1648 - 1715) upon reading Newton's work although Raphson only used it on polynomial equations to find their real roots.

The Newton-Raphson method is the true bridge between algebra (solving equations of the form $f(x) = 0$ and factoring) and geometry (finding tangent lines to the graph of $y = f(x)$). What follows will explore the idea of the Newton-Raphson Method and how tangent lines will help us solve equations both quickly and easily although not for exact solutions, only approximate ones.

The reason that we are studying the Newton-Raphson Method in this book is that it can also solve square non-linear systems of equations using matrices and their inverses as we shall see later. It is part of the wonderful effectiveness of the Newton-Raphson Method in that it can solve either a single equation or a square system of equations for its real or complex solutions, but only

approximately although to as many decimal places as you want.

Now we will discuss the important application of using tangent lines to solve a single equation of the form $f(x) = 0$ for approximate solutions either real or complex.

Example 1.1.1. The best way to understand the simplicity of this method and its geometric basis is to look at an example. Let's say that we want to solve the equation

$$x^3 - 5x^2 + 3x + 5 = 0 \qquad (1.1)$$

for an approximate solution x. We can easily estimate where the real solutions are by finding the x-intercepts of the graph of $y = x^3 - 5x^2 + 3x + 5$. Remember that the total number of real or complex roots to any polynomial is its degree (or order) which in this case is three. Also, when a polynomial has all real coefficients as this one does all of the complex roots (if there are any) occur in complex conjugate pairs. This particular polynomial has exactly three real roots and no complex roots by looking at its graph below.

> with(plots): with(plottools):

> f:= x -> x^3 - 5*x^2 + 3*x + 5:

> roots_f:= [fsolve(f(x), x, complex)];

$$roots\_f := [-0.7092753594, 1.806063434, 3.903211926]$$

> plot_xintercepts:= seq(circle([roots_f[j],0], .3, color = blue), j=1..3):

> plotf:= plot(f(x), x = -3..7, thickness = 2):

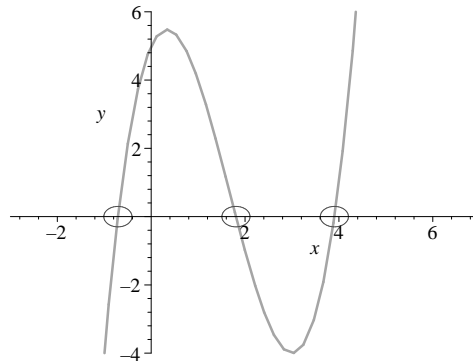> display({plotf, plot_xintercepts}, view = [-3..7,-4..6]);

Figure 1.1: The three roots of the polynomial $x^3 - 5x^2 + 3x + 5$ are its three x-intercepts (circled)

Figure 1.1 clearly shows that our equation has three real solutions, with a negative one near $x = -1$ and two positive ones near $x = 2$ and $x = 4$. Let's try to approximate the one near $x = 4$ as accurately as we can.

First, let's see what *fsolve* will give us for just the solution near $x = 4$. It has given us all the roots of this polynomial above if we use the complex option. *fsolve* uses many algorithms similar to and including the Newton-Raphson method in combination to approximate solutions both to a single equation or a square system of equations.

> fsolve(f(x), x, 3.5..4.5);

$$3.903211926$$

The idea that Raphson had was to take a value of x near $x = 4$, say $x = 5$, and put in the tangent line to the graph of our function $f(x)$ at $x = 5$. This tangent line goes through the point $(5, f(5)) = (5, 20)$ and has slope $\dfrac{df}{dx}(5)$, where $\dfrac{df}{dx}$ is the derivative function of $f(x)$. Now *Maple* can easily compute both $f(5)$ and $\dfrac{df}{dx}(5)$. The *unapply* command converts the expressions "diff(f(x), x)" into the derivative function $df(x)$.

> f(5);

$$20$$

> diff(f(x),x);

$$3x^2 - 10x + 3$$

> df:= unapply(diff(f(x),x),x);

$$df := x \rightarrow 3\,x^2 - 10\,x + 3$$

> df(5);

$$28$$

So the slope of the tangent line to the graph of $y = f(x)$ at the point $(5, 20)$ is 28 and the equation of this tangent line at $x = 5$ is $y - f(5) = \dfrac{df}{dx}(5)(x - 5)$ or $y = 28x - 120$. Let's now graph together this tangent line and the original function $f(x)$.

> plot_guess:= arrow([5,-10], [5,-4.5], .03, .15, .2, color = gold):
> plot_xint:= arrow([30/7,-10], [30/7,-4.5], .03, .15, .2, color = blue):
> plot_soln:= arrow([3.9,10], [3.9,2], .03, .15, .2, color = red):
> plotfwithtang:= plot([f(x), 28*x - 120], x = 3.5..5.5, color = [red, blue], thickness = [3,2]):

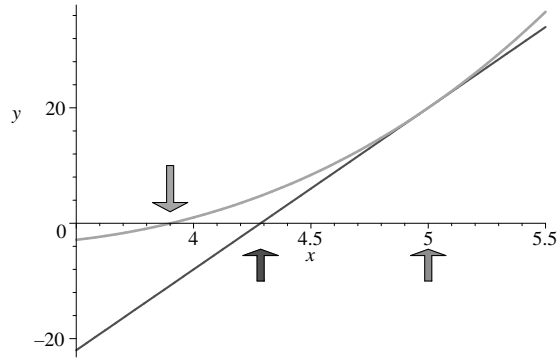> display({plot_guess, plotfwithtang, plot_xint, plot_soln});

Figure 1.2: Tangent line to our polynomial at $x = 5$

Upon inspection of Figure 1.2, note that the tangent line at $x = 5$ crosses the x-axis much closer to our solution near $x = 4$ than our very rough estimate of $x = 5$ for the placement of this tangent line. The x-intercept of this tangent line is the solution for x to the tangent line's equation $y = f(5) + \dfrac{df}{dx}(5)(x - 5) = 0$ which is

$$x = 5 - \frac{f(5)}{\dfrac{df}{dx}(5)}$$

$$= \frac{30}{7} \approx 4.285714286$$

(1.2)

So x-intercepts of tangent lines seem to move you closer to the x-intercepts of their function $f(x)$. This is what Raphson saw which Newton did not see because Newton forgot to look at the geometry of the situation and instead he concentrated on the algebra.

Raphson's next idea was to try to move even closer to the root of $f(x)$ (the x-intercept of $y = f(x)$ or solution to $f(x) = 0$) by repeating the tangent line, but now at the point given by this new value of $x = 4.285714286$ which is the x-intercept of the previous tangent line. Let's do it repeating the above work and see if Raphson was correct to do this. Let's now call our starting guess near the root by $x_0 = 5$ and the x-intercept of the tangent line at $x = 5$ by $x_1 = 4.285714286$. The value of x_1 is definitely closer to the root than the starting guess x_0.

> x0:= 5.:
> x1:= x0 - f(x0)/df(x0);

$$x1 := 4.285714286$$

> f(x1);

$$4.73760934$$

> df(x1);

$$15.24489796$$

> f(x1) - x1*df(x1);

$$-60.59766764$$

So the slope of the tangent line to the graph of $y = f(x)$ at the point $(x_1, f(x_1)) = (4.285714286, 4.73760934)$ is 15.24489796 and the equation of this tangent line at $x = x_1$ is

$$y - f(x_1) = \frac{df}{dx}(x_1)(x - x_1)$$

or

$$y = f(x_1) + \frac{df}{dx}(x_1)(x - x_1)$$
$$= 15.24489796x - 60.59766764$$

Let's now graph (see Figure 1.3) together this tangent line and the original function f(x).

> plot_xint2:= arrow([3.97495,-10], [3.97495,-4.5], .03, .15, .2, color = blue):
> plot_fwithtangs:= plot([f(x), 28*x - 120, 15.2449*x - 60.5977], x = 3.5..5.5, color = [red, blue, blue], thickness = [4,2,2]):
> display({plot_guess, plot_fwithtangs, plot_xint, plot_xint2, plot_soln}, view = [3.5..5.3, -15..30]);
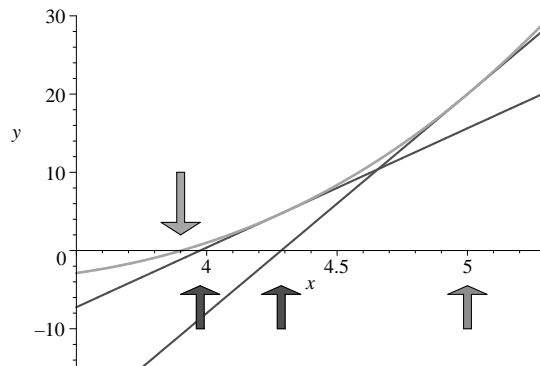
Figure 1.3: Tangent lines at $x = 5$ and $x = 4.28571$

Raphson clearly has a very good idea in the use of tangent lines to approximately solve single equations of the form $f(x) = 0$ if our picture above is truly correct. Happily for both Raphson and us, this picture is right and successive tangent lines and their x-intercepts do move closer and closer to the x-intercept of their underlying function $f(x)$.

The general formula for the x-intercept of the tangent line to the graph of $y = f(x)$ at the point where $x = a$ is

$$x = a - \frac{f(a)}{\dfrac{df}{dx}(a)}, \tag{1.3}$$

since the equation of the tangent line at $x = a$ is

$$y = f(a) + \frac{df}{dx}(a)(x - a)$$

If you now solve

$$f(a) + \frac{df}{dx}(a)(x - a) = 0$$

for x, you will get equation (1.3).

Let's use this formula below to get the successive x-intercepts for these tangent lines. The values of these x-intercepts are clearly moving towards the root of our polynomial which is roughly at $x = 3.9$ on the x-axis below the red arrow in the above plots.

The number of digits is now set to twelve instead of the default of ten since we want to get ten accurate digits when we round down to ten.

> Digits:= 12:

> x0:= 5.:

> x1:= x0 - f(x0)/df(x0);

$$x1 := 4.28571428571$$

> x2:= x1 - f(x1)/df(x1);

$$x2 := 3.97494740867$$

> x3:= x2 - f(x2)/df(x2);

$$x3 := 3.90652292594$$

> x4:= x3 - f(x3)/df(x3);

$$x4 := 3.90321950278$$

> x5:= x4 - f(x4)/df(x4);

$$x5 := 3.90321192597$$

> x6:= x5 - f(x5)/df(x5);

$$x6 := 3.90321192590$$

> x7:= x6 - f(x6)/df(x6);

$$x7 := 3.90321192592$$

This is the answer accurate to ten decimal places that *fsolve* gave us of $x = 3.903211926$. We perhaps got it the same way *fsolve* did since the Newton-Raphson Method is part of *fsolve*. To get this answer accurate to ten decimal places, it took only six repeated applications of the Newton-Raphson Method. This is a very fast procedure even when our starting guess of $x = 5$ is not very close to the root nearest to it.

It should be noted that you stop the Newton-Raphson Method when you get a repetition in the value for two consecutive x-intercepts of your tangent lines accurate to the number of digits you desire. This is the reason we could could stop after we got x_6 since $x_6 = x_5$ accurate to ten digits.

Newton's Method will in general solve equations of the form $f(x) = 0$ for the solution nearest a starting estimate of $x = x_0$. It then creates a list of values x_n where each x_n (the nth element of this list) is the x-intercept of the tangent line to $y = f(x)$ at the previous list value of $x = x_{n-1}$. This gives the general formula for the x_n of

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{df}{dx}(x_n)} \tag{1.4}$$

starting with x_0. Each x_n is usually closer to being a solution to $f(x) = 0$ than the previous x_{n-1}. The x_n can all be gotten from iterating the starting guess x_0 in the *iteration function*

$$g(x) = x - \frac{f(x)}{\frac{df}{dx}(x)} \tag{1.5}$$

This means that $x_{n+1} = g(x_n)$ for all n starting with 0.

The method we are using is called *iteration* since it begins with a starting guess value of x_0 and then finds the iteration values $x_1, x_2, \ldots$ thereafter using

the same formula $g(x)$ based on the previous value. The function $g(x)$ which computes these iteration values is called the *iterator*.

The following procedure will table some values of this Newton sequence of iterates from Example 1.1.1 and x_0's near our three solutions which you must provide as x_0. We will carry out $n = 8$ iterations of the Newton-Raphson Method to generate the list of lists called *newton*. Try different list of values for x_0 to see how they affect the Newton list moving toward the solutions nearest them. The starting list for x_0 below was chosen from the graphs above so that the tangents at these points will intersect the x-axis towards the solution nearest it. We stop iterating with the iterator $g(x)$ when two consecutive values in the sequence of x_n's are the same to the accuracy desired.

We will use the three starting guesses for x_0 of 0, 1, and 5 or the list $[0., 1., 5.]$ where the decimal point is used to tell *Maple* that we want decimal approximations and not exact values since to *Maple* any number with a decimal point in it is considered an approximation. Without these decimal points *Maple* would compute exact values giving us horrendous fractions for the values of our iterates instead of decimal approximations.

> Digits:= 12:

> f:= x -> x^3 - 5*x^2 + 3*x + 5:

> df:= unapply(diff(f(x),x), x);

$$df := x \to 3x^2 - 10x + 3$$

> x0:= [0., 1., 5.]:

> n := 8:

> g := unapply(x - f(x)/df(x), x);

$$g := x \to x - \frac{x^3 - 5x^2 + 3x + 5}{3x^2 - 10x + 3}$$

> newt:= proc(i) map(g, newt(i-1)) end:

> newt(1):= x0:

> newton:= [seq(newt(i), i = 1..n+1)];

$$[[0.0, 1.0, 5.0], [-1.66666666667, 2.0, 4.28571428571], [-1.00529100529,$$
$$1.80000000000, 3.97494740867], [-0.751331029986, 1.80606060606,$$
$$3.90652292594], [-0.710320317972, 1.80606343352, 3.90321950278],$$
$$[-0.709276029619, 1.80606343353, 3.90321192597], [-0.709275359437,$$
$$1.80606343355, 3.90321192590], [-0.709275359437, 1.80606343352,$$
$$3.90321192592], [-0.709275359437, 1.80606343353, 3.90321192589]]$$

> evalf(fsolve(f(x),x,complex),10);

$$-0.7092753594, 1.806063434, 3.903211926$$

> evalf(newton[9],10);

$$[-0.7092753594, 1.806063434, 3.903211926]$$

The *evalf* command has rounded the values in the list *newton*[9] to ten digits and it agrees to ten digits the results from *fsolve*. You have just seen the Newton-Raphson Method solve for the three real roots of our polynomial $f(x)$ simultaneously working on finding each root based on three different starting guesses near them.

The next section will continue looking at more examples of the uses of the Newton-Raphson algorithm. It will look for complex roots to polynomials, solve for thirteenth roots of numbers, find inverse trignometric function values.

Before we conclude this section, let's animate the tangent lines to our cubic polynomial $f(x)$ and see them moving along the polynomial's graph. This might help you understand why the Newton-Raphson Method works geometrically if you watch where these tangent lines' x-intercepts are going. We will plot 101 tangent lines for equally spaced points from $x = -2$ to $x = 6$. Figure 1.4 shows nine of the tangent lines from the animation.

> f:= x -> xˆ3 - 5*xˆ2 + 3*x + 5:
> plotf:= plot(f(x), x = -2.5..6.5, thickness = 3, color = red, view = -50..70):
> df:= unapply(diff(f(x),x), x):
> ptsoftangency:= [seq(-2. + (6+2)/100*(K-1),K=1..101)]:
> tangentlines:= [seq(f(ptsoftangency[K]) + df(ptsoftangency[K])*(x - ptsoftangency[K]), K = 1..101)]:
> plotsoftanglines:= [seq(plot(tangentlines[K], x=-2.5..6.5, thickness = 2, color = blue, view = -50..70), K=1..101)]:
> display([seq(display(plotf, plotsoftanglines[K]), K = 1..101)], insequence = true);
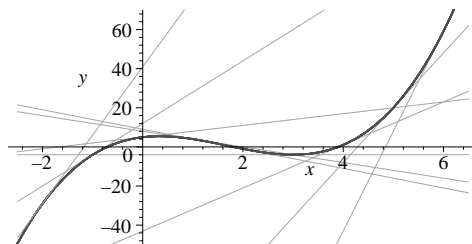

Figure 1.4: Nine frames in the tangent line animation, at $x = -2, -1, \ldots, 5, 6$

1.2   Examples of the Newton-Raphson Method

In this section, we will use the machinery developed in Section 1.1, we will apply the Newton-Raphson Method to specific problems.

Example 1.2.1. As example of the power of Newton's method, we will use it to find all the roots of the polynomial

$$8x^5 - 3x^4 + 2x^3 + 9x - 5 \tag{1.6}$$

both real and complex, and then to use these five roots to factor this polynomial completely. Since this polynomial has all real coefficients, the complex roots occur in complex conjugate pairs. This fact about the roots usually means we only need find roughly half as many roots as the degree of the polynomial. If you graph this polynomial (Figures 1.5 and 1.6) you will see that it probably only has one real root.

Why does an odd degree polynomial with all real coefficients have to have at least one real root? It is because there are going to be an even number of complex roots since they occur in complex conjugate pairs while the total number of all the roots must be the degree of the polynomial which is odd.

```
> Digits:= 10:
> f:= x -> 8*x^5-3*x^4+2*x^3+9*x-5:
> with(plots):
> plot(f(x), x = -10..10);
```
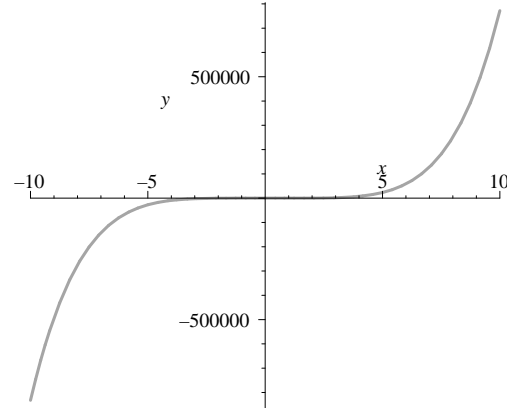

Figure 1.5: Graph of $f(x) = 8x^5 - 3x^4 + 2x^3 + 9x - 5$
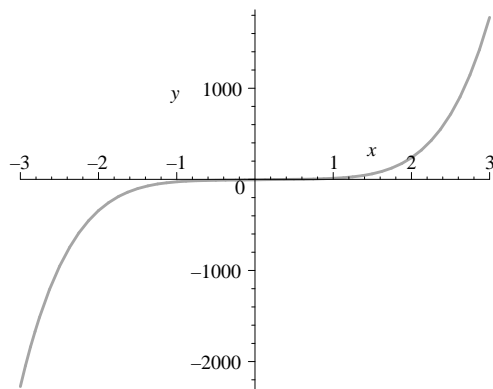
> plot(f(x), x = -3..3);

Figure 1.6: Graph of $f(x) = 8x^5 - 3x^4 + 2x^3 + 9x - 5$

> df:= unapply(diff(f(x),x),x);

$$df := x \rightarrow 40x^4 - 12x^3 + 6x^2 + 9$$

> x0:= [1., 1.+I, -1.-I]:
> g:= unapply(x - f(x)/df(x), x);

$$g := x \rightarrow x - \frac{8\,x^5 - 3\,x^4 + 2\,x^3 + 9\,x - 5}{40\,x^4 - 12\,x^3 + 6\,x^2 + 9}$$

> newt:= proc(i) map(g, newt(i-1)) end:
> newt(1):= x0:
> n:= 8:
> newton:= [seq(newt(i), i = 1..8)];

$newton := [\,[1., 1. + 1.I, -1. - 1.I], [0.7441860465, 0.8299022921$
$+ 0.8664659252I, -0.8350302309 - 0.8574919332I], [0.5697030175,$
$0.7080864702 + 0.8082690821I, -0.7475807128 - 0.7820854080I],$
$[0.5186129098, 0.6518047690 + 0.8166926208I, -0.7231360472$
$- 0.7605178310I], [0.5161116670, 0.6507190475 + 0.8253549920I,$
$- 0.7214092342 - 0.7588957152I], [0.5161066849, 0.6508476751$
$+ 0.8252171251I, -0.7214010980 - 0.7588870360I], [0.5161066849,$
$0.6508477553 + 0.8252171632I, -0.7214010978 - 0.7588870357I],$
$[0.5161066849, 0.6508477553 + 0.8252171633I, -0.7214010978$
$- 0.7588870357I]\,]$

> rootsf:= [op({op(newton[8]), op(map(conjugate, newton[8]))})];

$$rootsf := [0.5161066849, -0.7214010978 - 0.7588870357I, -0.7214010978$$
$$+ 0.7588870357I, 0.6508477553 - 0.8252171633I, 0.6508477553$$
$$+ 0.8252171633I]$$

> p:= coeff(f(x),x^5)*mul(x-rootsf[j], j=1..5);

$$p := 8(x - 0.5161066849)(x + 0.7214010978 + 0.7588870357I) \times$$
$$(x + 0.7214010978 - 0.7588870357I)(x - 0.6508477553+$$
$$0.8252171633I)(x - 0.6508477553 - 0.8252171633I)$$

> expand(p);

$$8x^5 - 2.999999999x^4 + (8.\ 10^{-10}I)x + 1.999999999x^3 - 1.6\ 10^{-9}x^2$$
$$+ 9.000000000x - 4.999999998 + 4.159135129\ 10^{-10})I$$

> f(x);
$$8x^5 - 3x^4 + 2x^3 + 9x - 5$$

> factor(f(x),complex);

$$8.(x + 0.7214010978 + 0.7588870357I)(x + 0.7214010978 - 0.7588870357I) \times$$
$$(x - 0.5161066849)(x - 0.6508477554 + 0.8252171634I) \times$$
$$(x - 0.6508477554 - 0.8252171634I)$$

The polynomial p above gives the complete factoring of $f(x)$ using the roots found from Newton's method. When we expand p we do not get $f(x)$ back exactly, but we are very close to it due to rounding error in our roots. The factor command in *Maple* using the complex option will factor the polynomial completely also.

■

Example 1.2.2. Now let's use the Newton-Raphson Method to solve the equation

$$e^x = \cos(2x) + 5 \tag{1.7}$$

for its single real solution where we actually solve

$$f(x) = e^x - \cos(2x) - 5 = 0$$

We will get that the solution is $x = 1.400545514$ after five iterations starting with $x_0 = 2$. In order to see where the solution to this equation lies, we will plot each side of the equation separately and find their intersection point (Figure 1.7).

> h:= x -> exp(x):
> k:= x -> cos(2*x)+5:
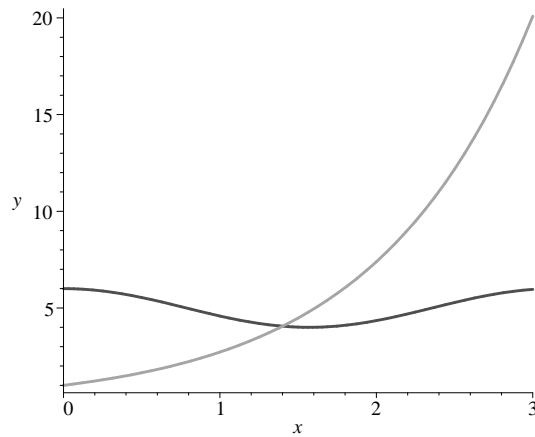> plot([h(x),k(x)], x = 0..3, color = [red,blue], thickness = 2);

Figure 1.7: The intersection of $h(x) = e^x$ and $k(x) = \cos(2x) + 5$ occurs close to $x = 1.5$

> f:= unapply(h(x) - k(x), x);

$$f := x \to x^x - \cos(2x) - 5$$

> df:= unapply(diff(f(x),x),x);

$$df := x \to e^x + 2\sin(2x)$$

> x0:= 2.:
> g:= unapply(x - f(x)/df(x), x);

$$g := x \to x - \frac{e^x - \cos(2x) - 5}{e^x + 2\sin(2x)}$$

> newt:= proc(i) g(newt(i-1)) end:
> newt(1):= x0:
> n:= 5:
> newton:= [seq(newt(i), i = 1..n+1)];

$$newton := [\, 2., 1.482133429, 1.400821839, 1.400545516,$$
$$1.400545514, 1.400545514\,]$$

Example 1.2.3. As our next example of the Newton-Raphson Method, we use it to find arctan(1.26195) from the function $\tan(x)$ alone. If we let $x = \arctan(1.26195)$, then by taking tangent of both sides we have

$$\tan(x) = \tan(\arctan(1.26195)) = 1.26195$$

Now rewriting this as $f(x) = \tan(x) - 1.26195 = 0$, we have our function $f(x)$ which has the arctan(1.26195) as root. Since $\arctan(z)$ is always between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$, we let $x_0 = 1$ because if $z > 0$, then $\arctan(z) > 0$. Using twelve digits, the end result of the Newton-Raphson Method should be accurate to ten digits.

> Digits:= 12:
> f:= x -> tan(x) - 1.26195:
> df:= unapply(diff(f(x),x),x);

$$df := x \rightarrow 1 + \tan(x)^2$$

> x0:= 1.:
> g:= unapply(x - f(x)/df(x), x);

$$g := x \rightarrow x - \frac{\tan(x) - 1.26195}{1 + (\tan(x))^2}$$

> newt:= proc(i) g(newt(i-1)) end:
> newt(1):= x0:
> n:= 5:
> newton:= [seq(newt(i), i = 1..6)];

$$newton := [\, 1., 0.913748036398, 0.900908331438, 0.900691798443,$$
$$0.900691739234, 0.900691739234\,]$$

> evalf(newton[6],10);
$$0.9006917392$$

> evalf(arctan(1.26195),10);

$$0.9006917392$$

Example 1.2.4. As the last example of this section, we use the Method to find $\sqrt[13]{8319407225}$, or $8319407225^{1/13}$. This thirteenth root can be found by letting $x = 8319407225^{1/13}$ and then rewriting this equation without the root as

$$x^{13} - 8319407225 = 0 \qquad\qquad (1.8)$$

We then have

$$f(x) = x^{13} - 8319407225 = 0 \qquad\qquad (1.9)$$

Now we have our function $f(x)$ for the Method. Since

$$5^{13} = 1220703125 < 8319407225 < 13060694016 = 6^{13}$$

we can take the starting guess to be either $x_0 = 5$ or $x_0 = 6$ since we usually choose an integer for the starting value x_0 (although not required). You could also plot $y = f(x)$ and look for the x-intercept of this graph.

```
> 5^13;
```
$$1220703125$$
```
> 6^13;
```
$$13060694016$$
```
> f:= x -> x^13-8319407225:
> df:= unapply(diff(f(x),x),x);
```
$$df := x \rightarrow 13\,x^{12}$$
```
> x0:= 6.:
> g:= unapply(x - f(x)/df(x), x);
```
$$g := x \rightarrow x - \frac{1}{13}\,\frac{x^{13} - 8319407225}{x^{12}}$$
```
> newt:= proc(i) g(newt(i-1)) end:
> newt(1):= x0:
> n:= 5:
> newton:= [seq(newt(i), i = 1..n+1)];
```
$$newton := [\,6., 5.83245253211, 5.79678752512, 5.79541014785,$$
$$5.79540818028, 5.79540818028\,]$$
```
> evalf(newton[6],10);
```
$$5.795408180$$
```
> evalf(8319407225^(1/13),10);
```
$$5.795408180$$

These last two examples of the Method illustrate that it can be used to find the values of inverse trignometric functions using the regular trignometric functions and to get roots of numbers using powers. It can also find the values of logarithms using exponentials. The Method also has a version which allows you to solve square systems of non-linear equations.

1.3   An Example of When the Newton-Raphson Method Does the Unexpected

The next example of the Method will show you that if you start with an x_0 which isn't very good, then you may slowly move towards a solution of $f(x) = 0$, but not the nearest one to where you started.

Example 1.3.1. Let's solve the trigonometric equation $\sin(x) = 0$ with a starting value of $x_0 = 1.97603146838$. You should expect that we will get the value of π using the Method. The graph of $\sin(x)$ is given in Figure 1.8 below.

> Digits:= 12:
> f:= x -> sin(x):
> plot(f(x), x=0..2*Pi);

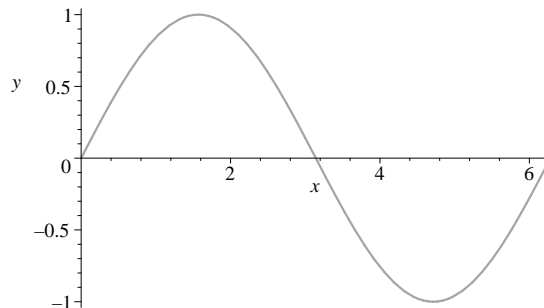Figure 1.8: Graph of $f(x) = \sin(x)$ on the interval $[0, 2\pi]$

> n:= 20:
> x0:= 1.97603146838:
> df:= unapply(diff(f(x),x),x);

$$df := x \rightarrow \cos(x)$$

> g:= unapply(x - f(x)/df(x), x);

$$g :=\rightarrow x - \frac{\sin(x)}{\cos(x)}$$

> newt:= proc(i) evalf(g(newt(i-1))) end:

> newt(1):= x0;

$$newt(1) := 1.97603146838$$

> newton:= [seq(newt(i), i = 1..n+1)];

$$newton := [\,1.97603146838, 4.30715383881, 1.97603146831, 4.30715383919,$$
$$1.97603146625, 4.30715385038, 1.97603140544, 4.30715418084,$$
$$1.97602960967, 4.30716393934, 1.97597657917, 4.30745215919,$$
$$1.97440902034, 4.31601007571, 1.92670276467, 4.61678041455,$$
$$-\,5.81064450865, -6.32181094255, -6.28316608667,$$
$$-\,6.28318530718, -6.28318530718\,]$$

> evalf(-2*Pi,12);

$$-6.28318530718$$

Now lets plot the first two tangent lines for Newton's method to $y = f(x) = \sin(x)$. It will explain what we see in this sequence of values which are moving towards -2π (see Figure 1.9).

> x1:= newt(2);

$$x1 := 4.30715383881$$

> plot([f(x), f(x0) + df(x0)*(x-x0), f(x1) + df(x1)*(x-x1)], x = 0..2*Pi, color = [red, blue, green], thickness = [3,2,2]);
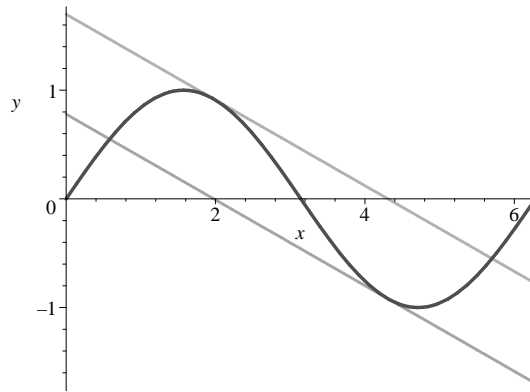

Figure 1.9: Graph of $f(x) = \sin(x)$ and parallel tangent lines

For this starting value of $x_0 = 1.97603146838$, the Newton sequence to twelve digits moves to -2π which is not the closest solution to $\sin(x) = 0$

to x_0, which is π. The values in this sequence approximately alternate each other for a long time until suddenly they move off towards -2π. They do this because the tangent lines used in the Method are approximately parallel lines where each has x-intercept approximately the x-coordinate value of the other. You should see what happens to this sequence of iterates if you increase the number of digits to sixteen.

The Newton-Raphson Method works quickly to give very good accuracy. It normally takes no more than five to ten iterations from a reasonable initial guess x_0 to get the solution accurate to ten or more decimal places. Each iteration usually produces greater accuracy, and you reach the accuracy you want when two consecutive iterations agree in value to this accuracy.

Chapter 2

The Newton-Raphson Method for Square Systems of Equations

2.1 Newton-Raphson for Two Equations in Two Unknowns

In this section we will discuss the Newton-Raphson method for solving square (as many equations as variables) systems of two non-linear equations

$$f(x, y) = 0, \ g(x, y) = 0$$

in the two variables x and y. In order to do this we must combine these two equations into a single equation of the form $F(x, y) = 0$ where F must give us a two component column vector and 0 is also the two component zero column vector, that is,

$$F(x, y) = \left[\begin{array}{c} f(x, y) \\ g(x, y) \end{array} \right] \tag{2.1}$$

and $0 = \left[\begin{array}{c} 0 \\ 0 \end{array} \right]$. Then clearly the equation $F(x, y) = 0$ is the same as

$$\left[\begin{array}{c} f(x, y) \\ g(x, y) \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \end{array} \right] \tag{2.2}$$

which is equivalent to the system of two equations $f(x, y) = 0$ and $g(x, y) = 0$.

Now let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous function which has continuous first partial derivatives where F is defined as in equation (2.1) for variables x, y and component functions $f(x, y)$, $g(x, y)$. We wish to solve the equation $F(x, y) =$

0 which is really solving simultaneously the square system of two non-linear equations given by $f(x, y) = 0$ and $g(x, y) = 0$.

In order to do this, we shall have to generalize the one variable Newton-Raphson method iterator formula for solving the equation $f(x) = 0$ given by the sequence p_k for p_0 the starting guess where

$$p_{k+1} = p_k - \frac{f(p_k)}{\frac{df}{dx}(p_k)} \tag{2.3}$$

or

$$p_{k+1} = g(p_k)$$

where

$$g(p_k) = p_k - \frac{f(p_k)}{\frac{df}{dx}(p_k)} \tag{2.4}$$

is the (single equation) Newton-Raphson iterator.

To see how this can be done, you must realize that now in the two equation case that

$$p_0 = \left[\begin{array}{c} x_0 \\ y_0 \end{array} \right] \tag{2.5}$$

is our starting guess as a point in the xy-plane written as a column vector,

$$p_k = \left[\begin{array}{c} x_k \\ y_k \end{array} \right] \tag{2.6}$$

is the kth iteration of our method, and

$$F(p_k) = \left[\begin{array}{c} f(x_k, y_k) \\ g(x_k, y_k) \end{array} \right]$$

are all two component column vectors in $\mathbb{R}^2$ and not numbers as in the single variable case. Thus, dividing a two component column vector by a derivative requires that we be dividing by a 2×2 matrix, or multiplying by the inverse of this matrix. Thus, we need to replace $\frac{df}{dx}$ in our old iterator $g(x)$ by a 2×2 matrix consisting of the four partial derivatives of $F(x, y)$, which are $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial g}{\partial x},$ and $\frac{\partial g}{\partial y}$.

The choice of this new derivative matrix is the 2×2 Jacobian matrix $J(x, y)$, of F(x,y), given by the 2×2 matrix

$$J(x, y) = \left[\begin{array}{cc} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{array} \right] \tag{2.7}$$

Other choices for this 2×2 matrix of partial derivatives are possible and you should see if they will also work in place of this Jacobian matrix, but this Jacobian matrix seems most logical if you think about the fact that for $F(x, y)$, the first row is $f(x, y)$ and $g(x, y)$ is in the second row while the variables are given as x first and y second in all these functions. Thus, the Newton-Raphson array (list or sequence) is now for a starting vector p_0, as defined in equation (2.5), given by

$$p_{k+1} = p_k - (J(p_k))^{-1} \, F(p_k) \tag{2.8}$$

where the vector $F(p_k)$ is multiplied on its left by the inverse of the Jacobian matrix $J(x, y)$ evaluated at p_k, i.e., $J(p_k) = J(x_k, y_k)$.

Note that finding $(J(p_k))^{-1}$ in each iteration is a formidable task if the system of equations is large, say 25×25 or more, and so at each iteration you can instead solve for p_{k+1} by solving the square linear system of equations

$$J(p_k) \, p_{k+1} = J(p_k) \, p_k - F(p_k)$$

where the components of p_{k+1} are the unknowns. Of course, it is easy to find p_{k+1} directly if you are in a small number of equations case such as ours using the inverse matrix.

Example 2.1.1. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be given in the form of equation (2.1), for

$$f(x, y) = \frac{1}{64}(x - 11)^2 - \frac{1}{100}(y - 7)^2 - 1, \; g(x, y) = (x - 3)^2 + (y - 1)^2 - 144$$

We wish to apply the Newton-Raphson method to solve $F(x, y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or equivalently the square system

$$f(x, y) = 0, \; g(x, y) = 0$$

The solutions are the intersection points of these two curves $f(x, y) = 0$ which is a hyperbola, and $g(x, y) = 0$ which is a circle, and is depicted in Figure 2.1.

```
> with(linalg): with(plots):
> f:= (x,y) -> (x-11)^2/64 - (y-7)^2/100 - 1:
> g:= (x,y) -> (x-3)^2 + (y-1)^2 - 400:
> plotf:= implicitplot(f(x,y) = 0, x = -25..25, y = -25..25, color = blue, grid
= [50,50]):
> plotg:= implicitplot(g(x,y) = 0, x = -25..25, y = -25..25, color = red, grid
= [50,50]):
```

> display({plotf, plotg}, scaling = constrained);

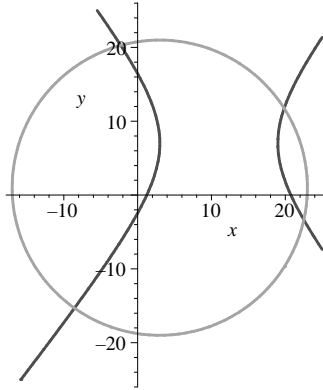

Figure 2.1: The hyperbola $f(x,y) = 0$ and circle $g(x,y) = 0$ intersect at four points.

It is clear from this plot of the circle and hyperbola that this system of equations has exactly four real solutions which are the intersection points of these two curves.

> F:= (x,y) -> vector([f(x,y), g(x,y)]):

> F(-2.,20);

$$\begin{bmatrix} -0.049375000 & -14. \end{bmatrix}$$

> jacobian(F(x,y), [x,y]);

$$\begin{bmatrix} \frac{1}{32}x - \frac{11}{32} & -\frac{1}{50}y + \frac{7}{50} \\ 2x - 6 & 2y - 2 \end{bmatrix}$$

The system iterator for Newton-Raphson will be called G and it is a function of x and y which outputs a two element list instead of a two component column vector. This was done in order to make the use of G the most convenient when iterating.

The iterates p_k are given by $newt[k]$ with starting guess p_0 given by $newt[0]$. In this case, since we graphed the two equations, we can find our starting guesses for the four solutions from this graph. Without a graph, you would have to plug guesses into $F(x,y)$ until you got a result close to the zero column vector; graphing is so much faster if we have a machine to do it for us. The number of iterations we will do is n which we take to be six below.

\> G:= unapply(convert(simplify(evalm(vector([x, y]) - jacobian(F(x,y),[x,y])^(-1)&*F(x, y))), list), x, y);

$$G := (x, y) \rightarrow \left[\frac{1}{2} \frac{41\,x^2y - 137\,x^2 + 5599\,y - 96\,y^2 - 43039}{41\,xy - 137\,x - 323\,y + 611}, \right.$$
$$\left. -\frac{1}{2} \frac{-41\,xy^2 + 323\,y^2 + 200\,x^2 - 10391\,x + 109173}{41\,xy - 137\,x - 323\,y + 611} \right]$$

\> G(-.25+.433*I,.433-.75*I);

$$[-35.67946232 + 9.764850390I, -92.91549690 + 40.44912122I]$$

\> n:= 6:
\> newt[0]:= [-2., 20]:
\> for i from 1 to n do
     newt[i]:= G(newt[i-1][1], newt[i-1][2]);
  od:
\> seq(newt[i-1],i=1..n+1);

$$[-2., 20], [-2.305821206, 20.28794179], [-2.302041868, 20.28440766],$$
$$[-2.302041289, 20.28440713], [-2.302041292, 20.28440712],$$
$$[-2.302041294, 20.28440712], [-2.302041294, 20.28440712]$$

\> root1 := newt[6];

$$root1 := [-2.302041294, 20.28440712]$$

So the root of the system of equations closest to the point $(-2, 20)$ is $newt[6]$ which we have called *root1*. We also check this root using *fsolve* where we must tell it where to look in order to get back just this single root.

After each computation of the table *newt* we will unassign it so that it is cleared to be used again without worrying that if we change n to a smaller value we may get leftover values in newt from its previous use. You can clear *newt* using the *unassign* command or by saying "newt:= 'newt';".

\> fsolve({f(x,y) = 0, g(x,y) = 0}, {x, y}, {x=-3..0,y=15..25});

$$\{x = -2.302041291, y = 20.28440712\}$$

\> unassign('newt');
\> n:= 7:
\> newt[0]:= [20., 13]:
\> for i from 1 to n do

```
        newt[i]:= G(newt[i-1][1], newt[i-1][2]);
    od:
> seq(newt[i-1],i=1..n+1);
```

$$[20., 13], [19.84349030, 11.84672207], [19.85957225, 11.75930869],$$
$$[19.85966012, 11.75880390], [19.85966010, 11.75880388],$$
$$[19.85966012, 11.75880390], [19.85966010, 11.75880388],$$
$$[19.85966012, 11.75880390]$$

```
> root2:= newt[7];
```

$$root2 := [19.85966012, 11.75880390]$$

```
> unassign('newt');
> newt[0]:= [20., -4]:
> newt[0]:= [20., 13]:
> for i from 1 to n do
        newt[i]:= G(newt[i-1][1], newt[i-1][2]);
    od:
> seq(newt[i-1],i=1..n+1);
```

$$[20., -4], [22.75576876, -3.230386204], [22.52089182, -3.359663132],$$
$$[22.51902586, -3.359774453], [22.51902576, -3.359774536],$$
$$[22.51902576, -3.359774542], [22.51902576, -3.359774539],$$
$$[22.51902574, -3.359774540]$$

```
> root3:= newt[6];
```

$$root3 := [22.51902576, -3.359774539]$$

```
> unassign('newt');
> newt[0]:= [-10., -16]:
> for i from 1 to n do
        newt[i]:= G(newt[i-1][1], newt[i-1][2]);
    od:
> seq(newt[i-1],i=1..n+1);
```

$$[-10., -16], [-8.625683855, -15.34506528], [-8.564570385, -15.31763460],$$
$$[-8.564449435, -15.31758282], [-8.564449440, -15.31758282],$$
$$[-8.564449445, -15.31758282], [-8.564449435, -15.31758282],$$
$$[-8.564449440, -15.31758282]$$

> root4:= newt[6];

$$root4 := [-8.564449435, -15.31758282]$$

Now that we have all four real roots of this system, we can plot these four points with the hyperbola and circle to see that they are the correct intersection points (see Figure 2.2).

> with(plottools):
> circle_root1:= circle(root1, 1):
> circle_root2:= circle(root2, 1):
> circle_root3:= circle(root3, 1):
> circle_root4:= circle(root4, 1):
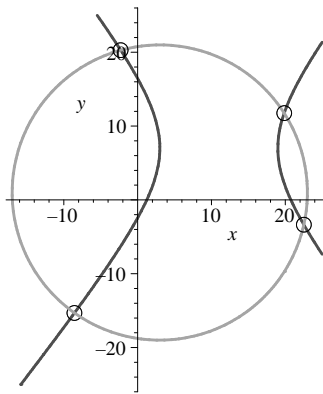> display({plotf, plotg, circle_root1, circle_root2, circle_root3, circle_root4}, scaling = constrained);

Figure 2.2: Intersection of the hyperbola $f(x, y) = 0$ and circle $g(x, y) = 0$ are the four points found by Newton-Raphson.

Example 2.1.2. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be given in the form of equation (2.1), for

$$f(x, y) = 3x^2y - y^3 + 5x - 8, \; g(x, y) = 3xy^2 - x^3 - 4y + 2$$

We wish to apply Newton's method to solve $F(x, y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or equivalently the square system

$$f(x, y) = 0, \; g(x, y) = 0$$

The real solutions are the real intersection points of these two curves.

> with(linalg): with(plots):

> f:= (x,y) -> 3*x^2*y-y^3+5*x-8:

> g:= (x,y) -> 3*x*y^2-x^3-4*y+2:

> plotf:= implicitplot(f(x,y) = 0, x = -10..10, y = -10..10, color = blue, grid = [50,50]):

> plotg:= implicitplot(g(x,y) = 0, x = -10..10, y = -10..10, color = red, grid = [50,50]):
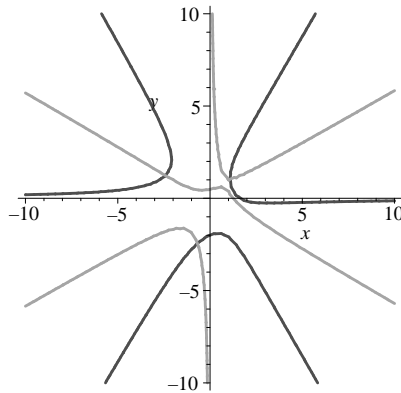
> display({plotf, plotg});

Figure 2.3: Intersection of the level curves $f(x, y) = 0$ and $g(x, y) = 0$

It is clear from Figure 2.3 that this system of equations has exactly three real solutions which are the intersection points of these two curves. How many total real and complex solutions are there to this system?

> F:= (x,y) -> vector([f(x,y), g(x,y)]):

> F(-2.+I, 5-3*I);

$$\left[\begin{array}{cc} 1. + 116.I & -22. + 229.I \end{array} \right]$$

> jacobian(F(x,y), [x,y]);

$$\left[\begin{array}{cc} 6\,xy + 5 & 3\,x^2 - 3\,y^2 \\ 3\,y^2 - 3\,x^2 & 6\,xy - 4 \end{array} \right]$$

> G:= unapply(convert(simplify(evalm(vector([x, y]) - jacobian(F(x,y),[x,y])^(-

1)&*F(x, y))), list), x, y);

$$G := (x, y) \rightarrow$$

$$\left[\frac{2\left(6\,x^3y^2 - 12\,x^2y + 3\,x^5 + 3\,xy^4 + 24\,xy + 4\,y^3 - 16 + 3\,x^2 - 3\,y^2\right)}{18\,x^2y^2 + 6\,xy - 20 + 9\,x^4 + 9\,y^4}, \right.$$

$$\left. \frac{2\left(6\,x^2y^3 + 15\,xy^2 + 3\,yx^4 + 3\,y^5 - 5\,x^3 + 12\,x^2 - 12\,y^2 - 6\,xy - 5\right)}{18\,x^2y^2 + 6\,xy - 20 + 9\,x^4 + 9\,y^4} \right]$$

> G(-.25+.433*I,.433-.75*I);

$$[1.067211700 - .3138214174I, -0.05450721154 - 0.4714997210I$$

> n:= 12:

> unassign('newt'):

> newt[0]:= [7.-10*I, -5+3*I]:

> for i from 1 to n do
    newt[i]:= G(newt[i-1][1], newt[i-1][2]);
  od:

> seq(newt[i-1],i=1..n+1);

$$[7. - 10.I, -5 + 3I],$$
$$[4.716076290 - 6.600715228I, -3.382974902 + 2.012179590I],$$
$$[3.224169736 - 4.308649374I, -2.326626794 + 1.372634953I],$$
$$[2.283146120 - 2.754133252I, -1.643936873 + 0.9907359528I],$$
$$[1.742859710 - 1.716145676I, -1.178525033 + 0.8220028622I],$$
$$[1.465295509 - 1.093767693I, -0.7709379106 + 0.7882814566I],$$
$$[1.292418266 - 0.7605470184I, -0.3991705530 + 0.7485523402I],$$
$$[1.204529252 - 0.5825119536I, -0.1451240359 + 0.6817317358I],$$
$$[1.196760919 - 0.5156573282I, -0.05289160786 + 0.6188832694I],$$
$$[1.202567805 - 0.5094748562I, -0.04975287894 + 0.6035737078I],$$
$$[1.202681506 - 0.5095861122I, -0.05002819636 + 0.6035124202I],$$
$$[1.202681462 - 0.5095860764I, -0.05002810388 + 0.6035124460I],$$
$$[1.202681463 - 0.5095860762I, -0.05002810432 + 0.6035124462I]$$

> F(newt[12][1], newt[12][2]);

$$\left[\; 7.\,10^{-9} + 1.\,10^{-9}I \quad -2.8\,10^{-9} + 3.\,10^{-9}I \; \right]$$

This last example indicates that we have at least one complex solution to this real system of equations.

Conclusions: (1) It should be clear from the above examples that even the system of equations version of the Newton-Raphson method is very fast! It also works to generate complex solutions as long as the starting value of $newt[0]$ is also complex when your equations are real. Redo the above example to see if you can find another complex solution. Is the complex conjugate of a solution in the last example also a solution?

(2) Also, rewrite the code above so that no inverse of the Jacobian is needed since this is impractical for large matrices and systems of equations.

(3) Now try using the Newton-Raphson method to solve the system

$$(x - 4)^2 + (y - 9)^2 = 25, \ (x - 3)^2 + (y - 7)^2 = 36$$

for both solutions. The two real solutions here are the intersection points of these two circles. Are there any complex solutions?

2.2   Newton-Raphson for Three Equations in Three Unknowns

In this section we will look at an example of using the system version of the Newton-Raphson method to solve a 3×3 system of equations which are all spheres in space.

Example 2.2.1. Our three spheres have the equations

$$\begin{aligned}
(x - 5)^2 + (y - 9)^2 + (z - 4)^2 &= 49 \\
(x - 2)^2 + (y - 7)^2 + (z - 13)^2 &= 100 \\
(x - 6)^2 + (y - 11)^2 + (z - 10)^2 &= 64
\end{aligned} \tag{2.9}$$

Let's now plot all three spheres and see if we can find any real intersection points of all three spheres (see Figure 2.4).

```
> with(linalg): with(plots):
> f:= (x,y,z) -> (x-5)^2 + (y-9)^2 + (z-4)^2 - 49:
> g:= (x,y,z) -> (x-2)^2 + (y-7)^2 + (z-13)^2 - 100:
> h:= (x,y,z) -> (x-6)^2 + (y-11)^2 + (z-10)^2 - 64:
> plotf:= implicitplot3d(f(x,y,z) = 0, x = -25..25, y = -6..25, z = -6..25, color
= blue, grid=[20,20,20]):
> plotg:= implicitplot3d(g(x,y,z) = 0, x = -25..25, y = -6..25, z = -6..25, color
= red, grid=[20,20,20]):
```

> ploth:= implicitplot3d(h(x,y,z) = 0, x = -25..25, y = -6..25, z = -6..25, color = green, grid=[20,20,20]):

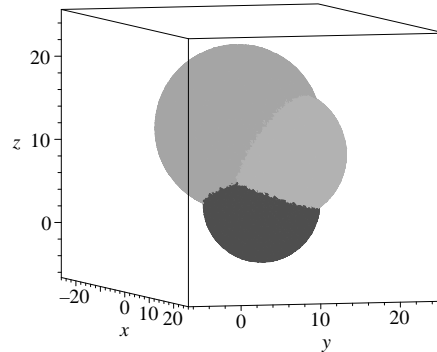> display({plotf, plotg, ploth}, axes = boxed, scaling = constrained);

Figure 2.4: Intersection of three spheres

It is clear from this plot of the three spheres that this system of equations has exactly two real solutions which are the intersection points of these three spheres near the points $(-2, 14, 6)$ and $(8, 5, 7)$.

We now use the Newton-Raphson system method to solve our problem where $F : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$F(x, y, z) = \begin{bmatrix} f(x, y, z) \\ g(x, y, z) \\ h(x, y, z) \end{bmatrix} \tag{2.10}$$

for

$$\begin{aligned} f(x, y, z) &= (x - 5)^2 + (y - 9)^2 + (z - 4)^2 - 49 \\ g(x, y, z) &= (x - 2)^2 + (y - 7)^2 + (z - 13)^2 - 100 \\ h(x, y, z) &= (x - 6)^2 + (y - 11)^2 + (z - 10)^2 - 64 \end{aligned} \tag{2.11}$$

We wish to apply the Newton-Raphson method to solve

$$F(x, y, z) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

> F:= (x,y,z) -> vector([f(x,y,z), g(x,y,z), h(x,y,z)]):

> F(-2.,14,6);

$$\begin{bmatrix} 29. & 14. & 25. \end{bmatrix}$$

$>$ jacobian(F(x,y,z), [x,y,z]);

$$\begin{bmatrix} 2\,x - 10 & 2\,y - 18 & 2\,z - 8 \\ 2\,x - 4 & 2\,y - 14 & 2\,z - 26 \\ 2\,x - 12 & 2\,y - 22 & 2\,z - 20 \end{bmatrix}$$

The system iterator for Newton-Raphson will be called G and it is a function of x, y, and z which outputs a three element list instead of a three component column vector. This was done in order to make the use of G the most convenient when iterating.

The iterates p_k are given by $newt[k]$ with starting guess p_0 given by $newt[0]$. In this case, since we graphed the three equations, we can find our starting guesses for the two solutions from this graph. Without a graph, you would have to plug guesses into $F(x, y, z)$ until you got a result close to the zero column vector; graphing is so much faster. The number of iterations we will do is n which we take to be seven below.

$>$ G:= unapply(convert(simplify(evalm(vector([x, y, z]) - jacobian(F(x, y, z), [x, y, z])^ (-1)&*F(x, y, z))), list), x, y, z);

$$G := (x, y) \rightarrow \left[\frac{3118 - 169\,z - 393\,y + 15\,x^2 + 15\,y^2 + 15\,z^2}{30\,x - 27\,y + 4\,z + 77}, \right.$$
$$-\frac{1}{2}\frac{-409\,z - 786\,x + 27\,x^2 + 27\,y^2 + 27\,z^2 + 3595}{30\,x - 27\,y + 4\,z + 77},$$
$$\left. \frac{1}{2}\frac{1699 + 338\,x - 409\,y + 4\,x^2 + 4\,y^2 + 4\,z^2}{30\,x - 27\,y + 4\,z + 77} \right]$$

$>$ G(-2.,14,6);

$$[-0.4213649852, 13.47922848, 5.577151335]$$

$>$ n:= 7:

$>$ newt[0]:= [-2., 14, 6]:

$>$ for i from 1 to n do
    newt[i]:= G(newt[i-1][1], newt[i-1][2], newt[i-1][3]);
  od:

$>$ seq(newt[i-1],i=1..n+1);

$$[-2., 14, 6],$$
$$[-.4213649852, 13.47922848, 5.577151335],$$
$$[-.2622019058, 13.33598171, 5.598373075],$$
$$[-.2596155823, 13.33365402, 5.598717920],$$
$$[-.2596148955, 13.33365340, 5.598718015],$$

$$[-.2596148984, 13.33365341, 5.598718015],$$
$$[-.2596149018, 13.33365340, 5.598718015],$$
$$[-.2596148982, 13.33365340, 5.598718015]$$

> root1 := newt[7];

$$root1 := [-.2596148982, 13.33365340, 5.598718015]$$

> F(root1[1], root1[2], root1[3]);

$$\begin{bmatrix} -4.\,10^{-8} & -1.0\,10^{-7} & -3.\,10^{-8} \end{bmatrix}$$

> fsolve({f(x, y, z) = 0, g(x, y, z) = 0, h(x, y, z)=0}, {x, y, z}, {x=-4..0, y=10..18, z=0..12});

$$\{x = -.2596148966, y = 13.33365341, z = 5.598718014\}$$

So the root of the system of equations closest to the point $(-2, 14, 6)$ is *newt*[7] which we have called *root1*. We also checked this root in two ways by first plugging it back in the function F to see if we get very close to zero (which we do) and then by using *fsolve* where we must tell it where to look in order to get back just this single root.

After each computation of the table newt we will unassign it so that it is cleared to be used again without worrying that if we change n to a smaller value we may get leftover values in newt from its previous use. You can clear newt using the *unassign* command or by saying "newt:= 'newt';".

> unassign('newt');

> n:= 10:

> newt[0]:= [8., 5, 7]:

> for i from 1 to n do
    newt[i]:= G(newt[i-1][1], newt[i-1][2], newt[i-1][3]);
  od:

> seq(newt[i-1],i=1..n+1);

$$[8., 5, 7],$$
$$[9.714285714, 4.357142857, 6.928571430],$$
$$[9.533470118, 4.519876898, 6.904462680],$$
$$[9.530132750, 4.522880526, 6.904017705],$$
$$[9.530131613, 4.522881542, 6.904017550],$$
$$[9.530131612, 4.522881554, 6.904017545],$$

$$[9.530131611, 4.522881541, 6.904017545],$$
$$[9.530131612, 4.522881544, 6.904017550],$$
$$[9.530131612, 4.522881538, 6.904017550],$$
$$[9.530131612, 4.522881536, 6.904017545],$$
$$[9.530131612, 4.522881542, 6.904017550]$$

> root2:= newt[10];
$$root2 := [9.530131612, 4.522881542, 6.904017550]$$

> F(root2[1], root2[2], root2[3]);

$$\begin{bmatrix} 4.\,10^{-8} & -3.\,10^{-7} & 5.\,10^{-8} \end{bmatrix}$$

> fsolve({f(x,y,z) = 0, g(x,y,z) = 0, h(x,y,z) = 0}, {x, y, z}, {x=5..15, y=0..10, z=0..14});

$$\{x = 9.530131614, y = 4.522881547, z = 6.904017549\}$$

Now we will start with a complex guess to see if our system might have any complex solutions.

> unassign('newt');
> n:= 10:
> newt[0]:= [15.-20*I, -14+8*I, -50-9*I]:
> for i from 1 to n do
    newt[i]:= G(newt[i-1][1], newt[i-1][2], newt[i-1][3]);
  od:
> seq(newt[i-1],i=1..n+1);

$$[15. - 20.I, -14 + 8I, -50 - 9I], [30.72301499 + 36.44398407I,$$
$$- 14.55071349 - 32.79958566I, 9.729735330 + 4.859197876I],$$
$$[17.83472141 + 18.00464381I, -2.951249272 - 16.20417943I,$$
$$8.011296190 + 2.400619174I], [11.55226588 + 8.569543596I,$$
$$2.702960707 - 7.712589245I, 7.173635450 + 1.142605814I],$$
$$[8.777004113 + 3.438297071I, 5.200696300 - 3.094467366I,$$
$$6.803600550 + .4584396097I], [8.418505686 + .2976097941I,$$
$$5.523344885 - 0.2678488144I, 6.755800760 + 0.03968130744I],$$
$$[9.673970474 - 0.09876135077I, 4.393426578 + 0.08888521560I,$$
$$6.923196065 - 0.01316818010I], [9.531271615 - 0.002796971041I,$$

$4.521855548 + 0.002517273967I, 6.904169550 - 0.0003729294942I],$

$[9.530130955 - 6.516297833\,10^{-7}I, 4.522882147 + 5.864707365\,10^{-7}I,$

$6.904017460 - 8.688510460\,10^{-8}I], [9.530131615 + 8.855537408\,10^{-14}I,$

$4.522881549 - 7.890932410\,10^{-14}I, 6.904017550 + 1.237092006\,10^{-14}I],$

$[9.530131616 - 2.761169677\,10^{-23}I, 4.522881547 + 3.876817403\,10^{-23}I,$

$6.904017545 + 2.647815558\,10^{-23}I]$

With this complex starting value we have gotten back to the previous real solution in *root2* since all of the imaginary parts of the solution above are all very close to zero. You should try more complex starting guesses and see if they all give these two real solutions or not. Do you believe that this system of equations has only two real solutions and no complex ones?