# Principles of Linear Algebra With *Maple*™
# Homework Problems
# Solutions Manual

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# Chapter 2

# Linear Systems of Equations and Matrices

## 2.1 Linear Systems of Equations

1. Give conditions on two lines such that their intersection results in a simultaneous solution that is (a) of dimension 0, (b) of dimension 1, (c) is empty.

(a) To be of dimension 0, the lines must not be parallel.

(b) To be of dimension 1, the two lines must be the same.

(c) To be of dimension 0, the lines must be parallel and distinct.

2. Give conditions on a line and a plane such that the intersection of these two geometric objects results in a simultaneous solution that is (a) of dimension 0, (b) of dimension 1, (c) is empty.

(a) To have an intersection of dimension 0, the line does not lie in the plane, or in a plane parallel to it.

(b) To have an intersection of dimension 1, the line must lie in the plane.

(c) To have an empty intersection, the line must lie in a plane parallel and distinct to the given plane.

3. Give conditions on two planes such that their intersection results in a simultaneous solution that is (a) of dimension 1, (b) of dimension 2, (c) is empty.

(a) Planes are not parallel and are distinct.

(b) Planes are the same.

(c) Planes are parallel and distinct.

4. Determine the dimension of the solutions to problem 2 in the *Maple* Problems section. You do not have to solve the systems by hand.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| a) | 0 | b) | 0 | c) | empty | d) | 1 |
| e) | 0 | f) | empty | g) | 0 | h) | 1 |
| i) | empty | | | | | | |

5. Solve by hand, for all of its intersection points (if any), each of the following linear systems of two lines:

(a) $3x - 2y = 9$
$5x + 4y = -13$

$\left\{x = \frac{5}{11}, y = -\frac{42}{11}\right\}$

(b) $x + 5y = 9$
$-2x - 10y = -2$

no solution

(c) $7x + 14y = -21$
$x + 2y = -3$

$\{x = -2y - 3, y = y\}$

6. Solve by hand, for all intersection points (if any), each of the following linear systems of three planes:

(a) $3x - 2y + z = 9$
$5x + 4y - 7z = -13$
$x - y - z = 2$

$\left\{x = \frac{20}{19}, y = -\frac{43}{19}, z = \frac{25}{19}\right\}$

(b) $x - 6y + 3z = 9$
$-2x + 4y - 7z = -2$
$-x - 2y - 4z = 15$

no solution

(c) $-x + 3y + 8z = 9$
$5x + 4y - 3z = -2$
$6x + y - 11z = -11$

$\left\{x = -\frac{42}{19} + \frac{41}{19}\,z, y = \frac{43}{19} - \frac{37}{19}\,z, z = z\right\}$

7. Show that $ax + by = c$ and $dx + ey = f$, for $a$ through $f$ real constants, are two parallel lines exactly when $ae - bd = 0$. As a consequence of this, these two lines intersect with dimension 0 exactly when $ae - bd \neq 0$.

Solving this system for $x$ and $y$ gives

$$\left\{x = \frac{ce - bf}{ae - db},\, y = \frac{af - dc}{ae - db}\right\}.$$

The above solution corresponds to all possible intersection points. Two lines have no intersection points if they are parallel and distinct, which corresponds to the above solution being empty. This occurs only when $ae - db = 0$. If $ae - db \neq 0$, there there exists only one point in the above solution set, hence the two lines intersect with dimension 0.

8. Show that $ax + by = c$ and $dx + ey = f$, for $a$ through $f$ real constants, are two parallel lines exactly when there is some real number $k$ where $dx + ey = k(ax + by)$.

If the two lines are parallel, then by problem 7, we know that $ae = db$. Thus, if we assume that $e \neq 0$, then we have $ax + by = c$ implies that $aex + bey = ce$. But $adb$, and therefore $aex + bey = ce$ is equivalent to $dbx + bey = ce$. Factoring and dividing by $b$ gives $dx + ey = \frac{ce}{b}$. Putting this all together, gives

$$\begin{aligned} ax + by &= \frac{1}{e}(aex + bey) \\ &= \frac{1}{e}(dbx + bey) \\ &= \frac{b}{e}(dx + ey). \end{aligned}$$

Therefore, setting $k = \frac{e}{b}$, gives $dx + ey = k(ax + by)$. If $e = 0$, then the problem becomes even easier,as this gives $dx = f$, a vertical line. We leave the details to the interested reader.

Now if we assume that $dx + ey = k(ax + by)$, then the two lines become:

$$\begin{aligned} ax + by &= c \\ ax + by &= \frac{f}{k}. \end{aligned}$$

These two lines are clearly parallel, since they have the same slope. Furthermore, the two lines are identical only if $f = ck$.

9. Show that $ax + by = c$ and $dx + ey = f$, for $a$ through $f$ real constants, has the single intersection point with coordinates $\left(\frac{ce - bf}{ae - bd}, \frac{af - cd}{ae - bd}\right)$, when $ae - bd \neq 0$.

Multiplying the first equation by $d$ and the second by $a$ gives

$$\begin{aligned} adx + bdy &= cd \\ adx + aey &= af, \end{aligned}$$

and then subtracting the first equation from the second yields

$$(ae - bd)y = af - cd \longrightarrow y = \frac{af - cd}{ae - bd}.$$

In a similar fashion, we take the original two equations, multiply the first by $e$, the second by $b$ and apply the same process to get $x = \dfrac{ce - bf}{ae - bd}$. Therefore, under the assumption that $ae - bd \neq 0$, we get the unique point $\left(\dfrac{ce - bf}{ae - bd}, \dfrac{af - cd}{ae - bd}\right)$ as the intersection point of the two lines.

10. Show that $ax + by = c$ and $dx + ey = f$, for $a$ through $f$ real constants, are two perpendicular lines exactly when $ad + be = 0$.

First we will assume that $ad \neq 0$ and $be \neq 0$. Then solving for $y$ in each of the equations gives

$$\begin{aligned} y &= -\frac{a}{b}x + \frac{c}{b} \\ y &= -\frac{d}{e}x + \frac{f}{e} \end{aligned}$$

If the two lines are perpendicular, then their slopes must be negative reciprocals, thus

$$-\frac{a}{b} = -\frac{1}{-\frac{d}{e}} = \frac{e}{d} \longrightarrow ad + be = 0.$$

These steps can be reversed as well.

Now if $ad = 0$, then $be = 0$ as well. If $a = 0$, then $b \neq 0$, otherwise, we would have the equation $0 = c$. This forces $e = 0$ and $d \neq 0$, giving the equations $by = c$ and $dx = f$. A similar arguments works for $ax = c$ and $ey = f$. Each of these pairs of lines are also perpendicular.

11. The definition of parallel planes is that they are either identical or they do not intersect. Show that $ax + by + cz = d$ and $ax + by + cz = e$, for $a$ through $e$ real constants, are two parallel planes.

Subtracting the two planes from each other gives the equation $0 = d - e$. This equation has no solution if $d \neq e$, therefore the planes are parallel and distinct. Any triplet $(x, y, z)$ will satisfy the equation if $d = e$, since the resulting equation is $0 = 0$. This, of course, is the case where both planes are the same.

12. Show that the two planes $ax + by + cz = d$ and $ex + fy + gz = h$, for $a$ through $h$ real constants, have the line of intersection given by

$$\left\{x = \frac{bg - cf}{af - be}z + \frac{df - bh}{af - be}, y = \frac{ce - ag}{af - be}z + \frac{ah - ed}{af - be}, z\right\}$$

for the independent variable $z$ when $af - be \neq 0$.

Just as in problem 9, we multiply by the appropriate constants to remove a variable. We will first remove $x$ by multiplying the first equation by $e$, the second by $a$ and taking their difference. This gives

$$(eb - af)y + (ce - ag)z = ed - ah \longrightarrow y = \frac{ce - ag}{af - be}z + \frac{ah - ed}{af - be}.$$

To solve for $x$ in terms of $z$, we need to get rid of the $y$'s. To do this, we multiply equation 1 by $f$, equation 2 by $b$ and take the difference:

$$(af - eb)x + (fc - bg)z = fd - bh \longrightarrow x = \frac{bg - cf}{af - be}z + \frac{df - bh}{af - be}.$$

Since $z$ is arbitrary we can express the line of intersection as the set of all points of the form:

$$\left\{x = \frac{bg - cf}{af - be}z + \frac{df - bh}{af - be}, y = \frac{ce - ag}{af - be}z + \frac{ah - ed}{af - be}, z\right\}.$$

## 2.2 Augmented Matrix of a Linear System and Row Operations

1. Given an augmented matrix, describe how to determine if it is in the simplest form possible for finding the solution to the system of equations from which the original augmented matrix was constructed.

In simplest form, every row of the matrix is required to be either all zeros or have a leading 1 in the row preceded by all zeros in its row. As you go down the rows of the matrix, the leading 1's of the matrix should move left to right with no two of them occurring in the same column. Furthermore, any row of all zeroes must be placed at the bottom of the simplest form matrix. As well, each column which contains a leading 1 must have all other entries of the column to be 0.

2. Given an augmented matrix, describe how to determine upon inspection whether or not the system of linear equations it represents has no solution, or an infinite number of solutions.

If the system has no solution, the simplest form matrix should have a row in which all entries are zero $a$ except for the entry in the last column, which is nonzero. This will result in an equation of the form $0 = a$, which is a contradiction for $a \neq 0$.

If the system has an infinite number of solutions, then the simplest form matrix must have at least one row where there is a leading followed by at least two columns which are nonzero in the same row.

3. Explain how to find the equation of a line through two points $(x_1, y_1)$, and $(x_2, y_2)$ and a plane through three points $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, and $(x_3, y_3, z_3)$ using linear systems.

A line in standard form is given by $ax + by = c$. If the points $(x_1, y_1)$, and $(x_2, y_2)$ lie on the line, then we have

$$\begin{aligned} ax_1 + by_1 &= c \\ ax_2 + by_2 &= c. \end{aligned}$$

This gives two equations and 3 unknowns, $a$, $b$ and $c$. You can solve for $a$ and $b$ in terms of $c$ to get an answer. Notice the line is unique in the sense that $ax + by = c$ is the same as $d(ax + by) = dc$, for any nonzero constant $d$.

In a similar fashion, a plane is defined by the equation $ax + by + cz = d$, and we can solve the system of three equations

$$\begin{aligned} ax_1 + by_1 + cz_1 &= d \\ ax_2 + by_2 + cz_2 &= d, \end{aligned}$$

for $a$, $b$, and $c$ in terms of $d$.

4. Convert each of the following systems to matrix form and determine to what set the resulting matrix belongs: $\mathbb{R}^{n\times m}$ or $\mathbb{C}^{n\times m}$ for specific values of $m$ and $n$.

(a) $$\begin{aligned} 3x + 4y &= 4 \\ 4x + 3y &= 8 \end{aligned} \longrightarrow \begin{bmatrix} 3 & 4 & 4 \\ 4 & 3 & 8 \end{bmatrix} \in \mathbb{R}^{2\times 3}$$

(b) $$\begin{aligned} x + 4iy &= 1 \\ x + y &= i \end{aligned} \longrightarrow \begin{bmatrix} 1 & 4i & 1 \\ 1 & 1 & i \\ -4 & 1 & -4i \end{bmatrix} \in \mathbb{C}^{3\times 3}$$

(c) $$\begin{aligned} x + 4y &= 1 \\ x + 2y &= 1 \\ -4x + y &= -1 \end{aligned} \longrightarrow \begin{bmatrix} 1 & 4 & 1 \\ 1 & 2 & 1 \\ -4 & 1 & -1 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

(d) $$\begin{aligned} 3x - 5y + 6z &= 1 \\ 4x + 3y - 2z &= 2 \end{aligned} \longrightarrow \begin{bmatrix} 3 & -5 & 6 & 1 \\ 4 & 3 & -2 & 2 \end{bmatrix} \in \mathbb{R}^{2\times 4}$$

(e) $$\begin{aligned} 3ix - 5y + 6z &= 1 \\ 4x + 3iy - 2z &= 2 \\ x - y + 2iz &= -i \end{aligned} \longrightarrow \begin{bmatrix} 3i & -5 & 6 & 1 \\ 4 & 3i & -2 & 2 \\ 1 & -1 & 2i & -i \end{bmatrix} \in \mathbb{C}^{3\times 4}$$

(f) $$\begin{aligned} x + y + z &= 1 \\ x - y + z &= 1 \\ x - 2y + z &= 2 \end{aligned} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix} \in \mathbb{R}^{3\times 4}$$

(g) $$\begin{aligned} x + y + 3z &= 1 \\ x - 2y + z &= 3 \\ x - 2y + z &= -2 \end{aligned} \longrightarrow \begin{bmatrix} 1 & 1 & 3 & 1 \\ 1 & -2 & 1 & 3 \\ 1 & -2 & 1 & -2 \end{bmatrix} \in \mathbb{R}^{3\times 4}$$

(h) $$\begin{aligned} 3x - 5y + 6z + 6t &= 1 \\ 3x - 5y + 6z + 6t &= 1 \\ x - y + 2z + 2t &= -1 \end{aligned} \longrightarrow \begin{bmatrix} 3 & -5 & 6 & 6 & 1 \\ 4 & 3 & -2 & -3 & 2 \\ 1 & -1 & 2 & 2 & -1 \end{bmatrix} \in \mathbb{R}^{3\times 5}$$

(i) $$\begin{aligned} x + y + z + t &= i \\ x - y + z + t &= 1 \\ x - 2y + z + t &= 2 \\ x - 2y + z + 2t &= 2 \end{aligned} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & i \\ 1 & -1 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 & 2 \\ 1 & -2 & 1 & 2 & 2 \end{bmatrix} \in \mathbb{C}^{4\times 5}$$

5. Convert the following systems to matrix form and then reduce each to its final augmented form using the row operations discussed in this section. Explain each step and show the modified augmented matrix at each step.

(a) $$\begin{aligned} 2x - 3y &= -5 \\ 3x + 7y &= 4 \end{aligned}$$

augmented matrix:

$$\begin{bmatrix} 2 & -3 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

(b) $$\begin{aligned} x + y + z &= 6 \\ -x + y - z &= -2 \\ x + 2y - z &= 2 \end{aligned}$$

augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ -1 & 1 & -1 & -2 \\ 1 & 2 & -1 & 2 \end{bmatrix}$$

final augmented matrix:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

final augmented matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

(c) $3ix - 4y = -26 - 21i$
$7x + 5iy = -36 + 39i$

augmented matrix:

$$\begin{bmatrix} 3i & -4 & -26-21i \\ 7 & 5i & -36+39i \end{bmatrix}$$

final augmented matrix:

$$\begin{bmatrix} 1 & 0 & -3+2i \\ 0 & 1 & 5+3i \end{bmatrix}$$

(d) $2x + 3y - 4z = -5$
$4x - 3y + 9z = 13$
$-6x + 9y + z = -8$

augmented matrix:

$$\begin{bmatrix} 2 & 3 & -4 & -5 \\ 4 & -3 & 9 & 13 \\ -6 & 9 & 1 & -8 \end{bmatrix}$$

final augmented matrix:

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(e) $10x + 15y - 7z = -35$
$-\dfrac{4}{3}x + 5y - 14z = -1$
$35x - 35y + 7z = -96$

augmented matrix:

$$\begin{bmatrix} 10 & 15 & -7 & -35 \\ -\frac{4}{3} & 5 & -14 & -1 \\ 35 & -35 & 7 & -96 \end{bmatrix}$$

final augmented matrix:

$$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -\frac{1}{5} \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix}$$

(f) $x - y + z = 0$
$x + 3y - 2z = 4 - 2i$
$-3x + y - 2iz = 2 + 4i$

augmented matrix:

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 3 & -2 & 4-2i \\ -3 & 1 & -2i & 2+4i \end{bmatrix}$$

final augmented matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 1-i \\ 0 & 1 & 0 & 1+i \\ 0 & 0 & 1 & 2i \end{bmatrix}$$

6. Using the final augmented matrices from the previous exercise, express the solutions to the original systems in set notation using the variables defined.

(a) $\{x = -1,\, y = 1\}$

(b) $\{x = 1,\, y = 2,\, z = 3\}$

(c) $\{x = -3 + 2i,\, y = 5 + 3i\}$

(d) $\left\{x = \frac{1}{2},\, y = -\frac{2}{3},\, z = 1\right\}$

(e) $\left\{x = -3,\, y = -\frac{1}{5},\, z = \frac{2}{7}\right\}$

(f) $\{x = 1 - i,\, y = 1 + i,\, z = 2i\}$

7. If the following augmented matrices represented a system of equations now

in reduced form, express the solutions in set notation.

(a) $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

$\{x = 3,\ y = -6,\ z = 2\}$

(b) $\begin{bmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & -6 & 0 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix}$

$\{w = 1 - 3z,\ x = 6z,\ y = -2 - 2z\}$

c) $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

no solution

## 2.3 Some Matrix Arithmetic

1. Consider the following matrices:

$$A = \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & -3 \\ 2 & 7 \end{bmatrix} \qquad C = \begin{bmatrix} 9 & -6 \\ -8 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & -9 & 2 \\ -4 & 6 & 2 \\ -1 & 3 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 4 & -9 & 2 \\ 6 & -1 & 3 \\ -7 & 2 & 5 \end{bmatrix} \quad F = \begin{bmatrix} 6 & -3 & 9 \\ -3 & 2 & -8 \\ 5 & 3 & 4 \end{bmatrix}$$

Perform the following matrix operations:

(a) $A - 2B = \begin{bmatrix} -9 & 9 \\ -6 & -13 \end{bmatrix}$

(b) $5A - 3C = \begin{bmatrix} -22 & 33 \\ 14 & -1 \end{bmatrix}$

(c) $2A + 3B - 4C = \begin{bmatrix} -19 & 21 \\ 34 & 15 \end{bmatrix}$

(d) $3(A - B + C) = \begin{bmatrix} 15 & 0 \\ -36 & -12 \end{bmatrix}$

(e) $2B + 5C = \begin{bmatrix} 55 & -36 \\ -36 & 24 \end{bmatrix}$

(f) $D - 3F + 4E = \begin{bmatrix} -1 & -36 & -17 \\ 29 & -4 & 38 \\ -44 & 2 & 8 \end{bmatrix}$

(g) $6D + 3F = \begin{bmatrix} 24 & -63 & 39 \\ -33 & 42 & -12 \\ 9 & 27 & 12 \end{bmatrix}$

(h) $2E + 3F = \begin{bmatrix} 26 & -27 & 31 \\ 3 & 4 & -18 \\ 1 & 13 & 22 \end{bmatrix}$

(i) $6D - 4E + 2F = \begin{bmatrix} 2 & -24 & 22 \\ -54 & 44 & -16 \\ 32 & 16 & -12 \end{bmatrix}$

(j) $2(D - 3E) + 5F = \begin{bmatrix} 8 & 21 & 37 \\ -59 & 28 & -54 \\ 65 & 9 & -10 \end{bmatrix}$

(k) $B - 4C + 3A = \begin{bmatrix} -28 & 30 \\ 28 & 2 \end{bmatrix}$

(l) $2(6D - 5F) = \begin{bmatrix} -48 & -78 & -66 \\ -18 & 52 & 104 \\ -62 & 6 & -40 \end{bmatrix}$

2. Consider the following matrices:

$$A = \begin{bmatrix} 1 & -4 \\ -2 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -7 & 8 \\ -2 & 2 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & -1 \\ -2 & 9 \\ 6 & -2 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 2 & 5 \\ -1 & 3 & 5 \end{bmatrix} \quad E = \begin{bmatrix} 1 & -7 & 8 \\ -2 & 2 & 3 \\ -2 & 1 & 6 \\ -2 & 8 & -7 \end{bmatrix} \quad F = \begin{bmatrix} 4 & -1 \\ -2 & 9 \\ 8 & -2 \\ 6 & -3 \end{bmatrix}$$

Determine which of the following matrix multiplications can be performed:

(a) $AA$ - yes (b) $AB$ - yes (c) $AC$ - no (d) $CA$ - yes e) $BB$ - no

(f) $FD$ - no (g) $DA$ - no (h) $BC$ - yes (i) $CF$ - no j) $DB$ - no

3. Find all 12 possible combinations of matrices from problem 2 that will allow a matrix multiplication to be performed.

The following matrix multiplications, using the matrices $A$ through $F$ defined above, can be performed:

$$AA,\ AB,\ BC,\ BD,\ CA,\ CB,\ DC,\ DD,\ EC,\ ED,\ FA,\ FB$$

4. Find a value of $a$ for which the following two matrices satisfy $AB = BA$:

$$A = \begin{bmatrix} a & 3 \\ -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 \\ 2 & -2 \end{bmatrix}$$

First we perform the multiplications $AB$ and $BA$:

$$AB = \begin{bmatrix} a+6 & -3a-6 \\ 0 & 4 \end{bmatrix}, BA = \begin{bmatrix} a+6 & 0 \\ 2a+4 & 4 \end{bmatrix}.$$

These two matrices are equal if and only if each of their entries are equal, this gives us the system of equations:

$$a+6 = a+6, \ -3a-6 = 0, \ 0 = 2a+4, \ 4 = 4.$$

Notice that the first and last equations always hold, this reduces our system two two equations, but these are multiples of each other as well. The solution to either the second or third equations is $a = -2$.

5. Perform the following matrix multiplications:

(a) $\begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -2 & -4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ -12 & -16 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -2 & 3 \\ 8 & -1 & -7 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 7 & 2 \\ 8 & -5 \end{bmatrix} = \begin{bmatrix} 8 & -15 \\ -79 & 65 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ -5 \end{bmatrix} = \begin{bmatrix} -27 \end{bmatrix}$

(d) $\begin{bmatrix} -2 & 2 \\ 5 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 8 \\ -4 & 5 & 9 \end{bmatrix} = \begin{bmatrix} -6 & 10 & 2 \\ -13 & 10 & 58 \\ 3 & -5 & -1 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & -1 \\ 3 & 0 \\ -8 & 7 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} -2 & 0 & -1 \\ 5 & 3 & 5 \end{bmatrix} = \begin{bmatrix} -9 & -3 & -7 \\ -6 & 0 & -3 \\ 51 & 21 & 43 \\ 18 & 6 & 14 \end{bmatrix}$

(f) $\begin{bmatrix} -2 & 3 & 1 \\ -7 & -1 & 4 \\ 3 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 2 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 19 \\ -9 & 21 \\ 19 & 24 \end{bmatrix}$

6. Write the following systems of equations in the matrix form $AX = B$:

(a)
$$\begin{aligned} 2x - 3y &= 7 \\ 4x + 5y &= 2 \end{aligned} \qquad \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

(b)
$$\begin{aligned} 23x - 6y + 4z &= 2 \\ 14x + 6y - 5z &= 4 \\ -5x + 4y &= -1 \end{aligned} \qquad \begin{bmatrix} 23 & -6 & 4 \\ 14 & 6 & -5 \\ -5 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$$

(c)
$$\begin{aligned} 2w + 4y - 5z &= 6 \\ -w + x - 4y + 3z &= 7 \\ 8w - 2x - 7z &= 9 \end{aligned} \qquad \begin{bmatrix} 2 & 0 & 4 & -5 \\ -1 & 1 & -4 & 3 \\ 8 & -2 & 0 & -7 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 9 \end{bmatrix}$$

(d)
$$\begin{aligned} w - 5x + 3y + 7z &= -8 \\ -9w + 3x + 4y - 6z &= 3 \\ 5w - 8x + y - 5z &= 3 \\ 6w + x + 9y + 13z &= -16 \end{aligned} \qquad \begin{bmatrix} 1 & -5 & 3 & 7 \\ -9 & 3 & 4 & -6 \\ 5 & -8 & 1 & -5 \\ 6 & 1 & 9 & 13 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 3 \\ -16 \end{bmatrix}$$

7. Consider the following three matrices:

$$A = \begin{bmatrix} 2 & 0 & 4 \\ 5 & 1 & -2 \\ 1 & 6 & -7 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 1 & 2 \\ -6 & 4 & 1 \\ 7 & 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 4 & -2 & 9 \\ 5 & -8 & 0 \\ 2 & 1 & -1 \end{bmatrix}$$

Show the following hold (assuming $c \in \mathbb{R}$).

(a) $A(BC) = (AB)C$

$$\begin{aligned} A(BC) &= \begin{bmatrix} 2 & 0 & 4 \\ 5 & 1 & -2 \\ 1 & 6 & -7 \end{bmatrix} \left( \begin{bmatrix} 5 & -4 & -11 \\ -2 & -19 & -55 \\ 32 & -12 & 61 \end{bmatrix} \right) \\ &= \begin{bmatrix} 138 & -56 & 222 \\ -41 & -15 & -232 \\ -231 & -34 & -768 \end{bmatrix} \\ &= \left( \begin{bmatrix} 26 & 2 & 12 \\ -25 & 9 & 7 \\ -86 & 25 & -6 \end{bmatrix} \right) \begin{bmatrix} 4 & -2 & 9 \\ 5 & -8 & 0 \\ 2 & 1 & -1 \end{bmatrix} \\ &= (AB)C \end{aligned}$$

(b) $A(B+C) = AB + AC$

$$\begin{aligned} A(B+C) &= \begin{bmatrix} 2 & 0 & 4 \\ 5 & 1 & -2 \\ 1 & 6 & -7 \end{bmatrix} \left( \begin{bmatrix} 3 & -1 & 11 \\ -1 & -4 & 1 \\ 9 & 1 & 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 42 & 2 & 26 \\ -4 & -11 & 54 \\ -66 & -32 & 10 \end{bmatrix} \\ &= \begin{bmatrix} 26 & 2 & 12 \\ -25 & 9 & 7 \\ -86 & 25 & -6 \end{bmatrix} + \begin{bmatrix} 16 & 0 & 14 \\ 21 & -20 & 47 \\ 20 & -57 & 16 \end{bmatrix} \\ &= AB + AC \end{aligned}$$

(c) $(A+B)C = AC + BC$

$$\begin{aligned} (A+B)C &= \left( \begin{bmatrix} 1 & 1 & 6 \\ -1 & 5 & -1 \\ 8 & 6 & -5 \end{bmatrix} \right) \begin{bmatrix} 4 & -2 & 9 \\ 5 & -8 & 0 \\ 2 & 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 21 & -4 & 3 \\ 19 & -39 & -8 \\ 52 & -69 & 77 \end{bmatrix} \\ &= \begin{bmatrix} 16 & 0 & 14 \\ 21 & -20 & 47 \\ 20 & -57 & 16 \end{bmatrix} + \begin{bmatrix} 5 & -4 & -11 \\ -2 & -19 & -55 \\ 32 & -12 & 61 \end{bmatrix} \\ &= AC + BC \end{aligned}$$

(d) $c(AB) = (cA)B$

$$\begin{aligned} c(AB) &= c \begin{bmatrix} 26 & 2 & 12 \\ -25 & 9 & 7 \\ -86 & 25 & -6 \end{bmatrix} \\ &= \begin{bmatrix} 26\,c & 2\,c & 12\,c \\ -25\,c & 9\,c & 7\,c \\ -86\,c & 25\,c & -6\,c \end{bmatrix} \\ &= \left( \begin{bmatrix} 2\,c & 0 & 4\,c \\ 5\,c & 1\,c & -2\,c \\ 1\,c & 6\,c & -7\,c \end{bmatrix} \right) \begin{bmatrix} -1 & 1 & 2 \\ -6 & 4 & 1 \\ 7 & 0 & 2 \end{bmatrix} \\ &= (cA)B \end{aligned}$$

(e) $A(cB) = (Ac)B$

$$\begin{aligned} A(cB) &= \begin{bmatrix} 2 & 0 & 4 \\ 5 & 1 & -2 \\ 1 & 6 & -7 \end{bmatrix} \left( \begin{bmatrix} -1\,c & 1\,c & 2\,c \\ -6\,c & 4\,c & 1\,c \\ 7\,c & 0 & 2\,c \end{bmatrix} \right) \\ &= \begin{bmatrix} 26\,c & 2\,c & 12\,c \\ -25\,c & 9\,c & 7\,c \\ -86\,c & 25\,c & -6\,c \end{bmatrix} \\ &= \left( \begin{bmatrix} 2\,c & 0 & 4\,c \\ 5\,c & 1\,c & -2\,c \\ 1\,c & 6\,c & -7\,c \end{bmatrix} \right) \begin{bmatrix} -1 & 1 & 2 \\ -6 & 4 & 1 \\ 7 & 0 & 2 \end{bmatrix} \\ &= (Ac)B \end{aligned}$$

(f) $(AB)c = A(Bc)$

$$\begin{aligned} (AB)c &= \begin{bmatrix} 26 & 2 & 12 \\ -25 & 9 & 7 \\ -86 & 25 & -6 \end{bmatrix} c \\ &= \begin{bmatrix} 26\,c & 2\,c & 12\,c \\ -25\,c & 9\,c & 7\,c \\ -86\,c & 25\,c & -6\,c \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 4 \\ 5 & 1 & -2 \\ 1 & 6 & -7 \end{bmatrix} \left( \begin{bmatrix} -1\,c & 1\,c & 2\,c \\ -6\,c & 4\,c & 1\,c \\ 7\,c & 0 & 2\,c \end{bmatrix} \right) \\ &= A(Bc) \end{aligned}$$

8. For part (a) of problem 7, what are the general dimensions of $A$, $B$, and $C$ such that one can perform the matrix multiplication? Do the same for the matrices of parts (b) and (c).

(a) The most general dimensions are $A : k \times m$, $B : m \times n$ and $C : n \times m$, for $k$, $m$ and $n$ arbitrary positive integers.

(b) The only difference between this part and part (a) is that $B$ and $C$ must be of the same dimension, thus $A : k \times m$, $B : m \times n$ and $C : m \times n$.

(c) Similar to part (b) the solution here is $A$ and $B$ must be of the same dimension, thus $A : m \times n$, $B : m \times n$ and $C : n \times k$.

9. Let $c$ be a scalar and $A$ be any $2 \times 2$ matrix, show that $cA = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} A$. Is a similar statement true for any size square matrix $A$? Generalize this to any size matrix $A$ so that general scalar multiplication is turned into matrix

multiplication by a diagonal matrix?

To show that $cA = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} A$, we simply recognize the fact that $\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} = cI_2$, for $I_2$ the $2 \times 2$ identity matrix. Therefore,

$$\begin{aligned} cA &= c(I_2 A) \\ &= (cI_2)A \\ &= \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} A. \end{aligned}$$

To generalize this process, we simply notice that if $A \in \mathbb{R}^{n \times m}$, then $I_n A = A$, for $I_n$ the $n \times n$ identity matrix. Therefore, applying the same process as above, we get

$$\begin{aligned} cA &= c(I_n A) \\ &= (cI_n)A \\ &= \begin{bmatrix} c & 0 & \cdots & 0 \\ 0 & c & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & c \end{bmatrix} A. \end{aligned}$$

10. Let $A$, $B$, $C$, and $D$ be four $2 \times 2$ matrices. Let $K = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and $L = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}$ be two block diagonal $4 \times 4$ matrices with $A$, $B$, $C$, and $D$ on their diagonals, the 0's in these two matrices are the zero $2 \times 2$ matrices. Show that $KL = \begin{bmatrix} AC & 0 \\ 0 & BD \end{bmatrix}$ and $K^n = \begin{bmatrix} A^n & 0 \\ 0 & B^n \end{bmatrix}$ for any positive integer $n$. Generalize this problem where there is no restriction on the sizes of $A$, $B$, $C$, and $D$ as long as they are square matrices, although you may want some of their sizes to be the same. Will this work for larger size block diagonal matrices?

First, we compute the upper left $2 \times 2$ block of the matrix $KL$. Notice that

$$(KL)_{i,j} = \sum_{k=1}^{4} K_{i,k} L_{k,j},$$

but $K_{i,k} = 0$ for $1 \leq i \leq 2$ and $3 \leq k \leq 4$, and similarly $K_{k,j} = 0$ for $1 \leq j \leq 2$ and $3 \leq k \leq 4$. Therefore, we have

$$(KL)_{i,j} = \sum_{k=1}^{2} K_{i,k} L_{k,j},$$

for $1 \leq i, j \leq 2$. But this is the definition of $AC_{i,j}$. A similar argument holds for the lower right $2 \times 2$ block.

Now, the off diagonal $2 \times 2$ blocks are zero for exactly the opposite reason. For instance, consider the upper right $2 \times 2$ block, which corresponds to $1 \leq i \leq 2$ and $3 \leq j \leq 4$:

$$\begin{aligned}
(KL)_{i,j} &= \sum_{k=1}^{4} K_{i,k} L_{k,j} \\
&= \sum_{k=1}^{2} K_{i,k} L_{k,j} + \sum_{k=3}^{4} K_{i,k} L_{k,j} \\
&= \sum_{k=1}^{2} K_{i,k} \cdot 0 + \sum_{k=3}^{4} 0 \cdot L_{k,j} \\
&= 0.
\end{aligned}$$

Upon setting $L = K$, we automatically arrive at $K^2 = \begin{bmatrix} A^2 & 0 \\ 0 & B^2 \end{bmatrix}$, which can therefore be generalized to $K^n = \begin{bmatrix} A^n & 0 \\ 0 & B^n \end{bmatrix}$ for any positive integer $n$.

This process can be generalized to non-square blocks as long as the dimensions of the blocks of the first matrix are equal to the dimensions of the transpose of the second matrix, so that zero blocks still occur off the diagonal.

11. Define the set $\mathbf{M}$ as follows:

$$\mathbf{M} = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$$

Show that the following are true.

(a) For any two $K, L \in \mathbf{M}$, $KL = LK$, that is, matrix multiplication in $\mathbf{M}$ is commutative.

Setting $K = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ and $L = \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$, it is easy to compute that

$$KL = \begin{bmatrix} ac - bd & ad + bc \\ -bc - ad & ac - bd \end{bmatrix} = LK.$$

(b) Show that

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & 0 \\ 0 & a^2 + b^2 \end{bmatrix} = (a^2 + b^2) I_2,$$

where $I_2$ is the $2 \times 2$ identity matrix.

This is a straightforward computation involving two $2 \times 2$ matrices.

(c) Show that

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_2$$

Hence $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is the square root of $-I_2$.

$$\begin{aligned} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^2 &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= -I_2 \end{aligned}$$

(d) Do these properties of $\mathbf{M}$ remind you of any other set and its properties?

12. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$. Let $C = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, with $ad-bc \neq 0$. Show that $AC = I_2$ and $CA = I_2$ . What does this make $C$ with respect to $A$?

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right) &= \frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & -bc+ad \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I_2. \end{aligned}$$

The argument to show that $CA = I_2$ is similar. What this means is that $C$ is $A$'s multiplicative inverse, i.e. $C = A^{-1}$.

13. Use the result from problem 12 to solve the linear system

$$\begin{aligned} 5x - 7y &= 11 \\ 9x + 2y &= -4 \end{aligned}$$

Check your answer by another means.

First, we rewrite the system in matrix form as:

$$\begin{bmatrix} 5 & -7 \\ 9 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \end{bmatrix}.$$

Setting $A = \begin{bmatrix} 5 & -7 \\ 9 & 2 \end{bmatrix}$, then

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} 11 \\ -4 \end{bmatrix}.$$

Using the formula from the previous problem gives

$$A^{-1} = \begin{bmatrix} \frac{2}{73} & \frac{7}{73} \\ -\frac{9}{73} & \frac{5}{73} \end{bmatrix},$$

and therefore,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{2}{73} & \frac{7}{73} \\ -\frac{9}{73} & \frac{5}{73} \end{bmatrix} \begin{bmatrix} 11 \\ -4 \end{bmatrix} = \begin{bmatrix} -\frac{6}{73} \\ -\frac{119}{73} \end{bmatrix}.$$

This gives the solution $x = -\frac{6}{73}$ and $y = -\frac{119}{73}$.

From the methods of this chapter, we could have also computed the *rref* matrix of the augmented matrix given by

$$\begin{bmatrix} 5 & -7 & 11 \\ 9 & 2 & -4 \end{bmatrix},$$

which, in row reduced form is

$$\begin{bmatrix} 1 & 0 & -\frac{6}{73} \\ 0 & 1 & -\frac{119}{73} \end{bmatrix}.$$

Notice that this also yields the solution $x = -\frac{6}{73}$ and $y = -\frac{119}{73}$.

14. Let your $2 \times 2$ linear system of equations be given as the matrix equation $AX = B$, where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, \text{ and } B = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

where $ad - bc \neq 0$. Using the result of problem 12, what is the solution formula for $X$?

The solution is given by $X = A^{-1}B$:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

# Chapter 3

# Gauss-Jordan Elimination and Reduced Row Echelon Form

## 3.1 Gauss-Jordan Elimination and *rref*

1. Give both an algebraic and geometric explanation of why underdetermined systems typically have an infinite number of solutions while overdetermined systems typically have no solution.

A system is underdetermined implies that there are more variables than equations. Therefore, when solving there will most likely be variables 'left over'. For instance, given $n$ equations and $n+k$ variables one can typically solve for the first $n$ variables in terms of the remaining $k$. Geometrically speaking, if we intersect two $n+k$ dimensional objects, the result is usually of dimension $n+k-1$. Repeating this process $n-1$ more times would give the intersection of $n$ of the $n+k$ dimensional objects, which we would expect to be of dimension $k$.

A system is overdetermined implies that there are fewer variables than equations. Algebraically, this allows for the possibility of isolating a single variable from several different unrelated equations, giving rise to the possibility of different solutions for a single variable, implying no solution exists. Geometrically speaking, it is easiest to think of lines in $\mathbb{R}^2$ first. A line has as its solution, itself. Two lines typically intersect at a single point. Three lines typically do not intersect at a common point. Applying the same geometric argument as in the underdetermined system, (but with swapping $n$ and $n+k$) gives a negative dimension to the solution, implying that no solution exists.

2. Using your answer to problem 1, construct (if possible) underdetermined and overdetermined systems that have a single solution, that is, a solution of dimension 0.

As an example of an overdetermined system, consider:

$$\begin{aligned} x + y &= 1 \\ x - 2y &= 3, \\ 2x - y &= 4 \end{aligned}$$

which is a system of three equations with only two unknown variables. The solution to this system is $\left\{\left(\frac{5}{3}, -\frac{2}{3}\right)\right\}$, and is thus of dimension 0.

Underdetermined systems have no solutions of dimension 0. For instance, consider equations two equation in $\mathbb{R}^3$. The intersection of two planes can be of no dimension, or dimension 1 or 2. For instance, the following system has no solution, thus has no dimension:

$$\begin{aligned} x + y + z &= 1 \\ x + y + z &= -1. \end{aligned}$$

The next system has a solution of dimension 1 (a line):

$$\begin{aligned} x + y + z &= 1 \\ x - y &= -1, \end{aligned}$$

and can be expressed as $\left\{\left(x = \frac{1}{2} - \frac{1}{2}z, y = \frac{1}{2} - \frac{1}{2}z, z\right)\right\}$.

Finally, if two planes are the same (up to multiplication by a scalar), the intersection is dimension 2. Consider the system

$$\begin{aligned} x + y + z &= 1 \\ 2x + 2y + 2z &= 1, \end{aligned}$$

whose solution is $\{(x = 1 - y - z, y, z)\}$. Notice that the variable $x$ depends on the two other variables, $y$ and $z$, which are independent. Therefore, the solution has dimension 2.

3. Perform Gauss-Jordan elimination on the following systems of equations, but with one restriction: You are not allowed to swap row 1 with any other. Be sure to show each step in the process.

(a) $\begin{array}{l} x-3y+z=6 \\ -2x+6y+3z=4 \\ 2x+5y+6z=1 \end{array} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -\frac{203}{55} \\ 0 & 1 & 0 & -\frac{119}{55} \\ 0 & 0 & 1 & \frac{16}{5} \end{bmatrix}$

step 1) 2row1 + row2 $\to$ row2 step 2) -2row1 + row3 $\to$ row3
step 3) row3 + row2 $\to$ row2 step 4) $\frac{1}{11}$row2 $\to$ row 2
step 5) 3row2 + row1 $\to$ row1 step 6) -11row2 +row3 $\to$ row3
step 7) $-\frac{1}{5}$row3 $\to$ row3 step 8) $-\frac{9}{11}$row3 + row2 $\to$ row2
step 9) $-\frac{38}{11}$row3 + row1 $\to$ row1

(b) $\begin{array}{l} 2x+3y-5z=7 \\ 3x+2y+7z=8 \\ 4x+6y+2z=1 \end{array} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{523}{60} \\ 0 & 1 & 0 & -\frac{317}{60} \\ 0 & 0 & 1 & -\frac{13}{12} \end{bmatrix}$

step 1) $\frac{1}{2}$row1 $\to$ row1 step 2) -3row1+row2 $\to$ row2
step 3) -4row1+row3 $\to$ row3 step 4) $-\frac{2}{5}$row2 $\to$ row2
step 5) $-\frac{3}{2}$row2 + row1 $\to$ row1 step 6) $\frac{1}{12}$row3 $\to$ row3
step 7) $\frac{29}{5}$row3 + row2 $\to$ row2 step 8) $-\frac{31}{5}$row3 + row1 $\to$ row1

(c) $\begin{array}{l} x+y+z=1 \\ 2w+2x+3y+z=4 \\ 2w+3x+4y+2z=5 \\ 4w+6x+8y+4z=10 \end{array} \longrightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

step 1) $\frac{1}{2}$row2+row1 $\to$ row1 step 2) -2row1+row2 $\to$ row2
step 3) -2row1+row3 $\to$ row3 step 4) -2row1+row4 $\to$ row4
step 5) $-\frac{1}{2}$row2 $\to$ row2 step 6) -2row2+row1 $\to$ row1
step 7) 2row2 + row4 $\to$ row4 step 8) row2+row3 $\to$ row3

(d) $\begin{array}{l} w+x+y+z=3 \\ 2w+2x+2y-3z=5 \\ -w+x+y+z=6 \\ 4w+6x+y-7z=8 \end{array} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & 0 & \frac{111}{50} \\ 0 & 0 & 1 & 0 & \frac{52}{25} \\ 0 & 0 & 0 & 1 & \frac{1}{5} \end{bmatrix}$

step 1) -2 row1 + row2 $\to$ row2 step 2) row1 + row3 $\to$ row3
step 3) -4row1 + row4 $\to$ row4 step 4) $\frac{1}{2}$row3 + row2 $\to$ row2
step 5) -row2 + row1 $\to$ row1 step 6) -2row2 + row3 $\to$ row3
step 7) -2row2 + row4 $\to$ row4 step 8) $-\frac{1}{5}$row4 + row3 $\to$ row3
step 9) 5row3 + row4 $\to$ row4 step 10) -row3 + row2 $\to$ row2
step 11) $\frac{1}{50}$row4 $\to$ row4 step 12) $-\frac{53}{5}$row4 + row3 $\to$ row3
step 13) $\frac{73}{5}$row4 +row2 $\to$ row2 step 14) -5row4 + row1 $\to$ row1

(e) $\begin{array}{l} 3x+4y-7z=1 \\ -2x+4y-8z=2 \\ 5x+z=-1 \end{array} \longrightarrow \begin{bmatrix} 1 & 0 & \frac{1}{5} & -\frac{1}{5} \\ 0 & 1 & -\frac{19}{10} & \frac{2}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

step 1) $\frac{1}{3}$row1 → row1 step 2) 2row1 +row2 → row2
step 3) -5row1+row3 → row3 step 4) $\frac{3}{20}$row2 → row2
step 5) $\frac{20}{3}$row2 + row3 → row3 step 6) -$\frac{4}{3}$row2 + row1 → row1

(f) $\begin{array}{l} 3x+4y-7z=1 \\ -2x+4y-8z=2 \\ 5x+z=-1 \\ -3x+4y+3z=2 \end{array} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -\frac{11}{56} \\ 0 & 1 & 0 & \frac{41}{112} \\ 0 & 0 & 1 & -\frac{1}{56} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

step 1) $\frac{1}{3}$row1 → row1 step 2) 2row1 +row2 → row2
step 3) -5row1+row3 → row3 step 4) 3row1 + row4 → row4
step 5) $\frac{3}{20}$row2 → row2 step 6) -8row2 + row4 → row4
step 7) $\frac{20}{3}$row2 + row3 → row3 step 8) $-\frac{4}{3}$row2 + row1 → row1
step 9) $\frac{5}{56}$row4 → row4 step 10) row4+row3 → row3
step 11) -row3 + row4 → row4 step 12) $\frac{19}{10}$row3 + row2 → row2
step 13) $-\frac{1}{5}$row3 + row1 → row1

4. Write out the solutions to each system from problem 3, and give the dimension of the solution and the space $\mathbb{R}^n$ it lies in.

(a) $\left\{x=-\frac{203}{55}, y=-\frac{119}{55}, z=\frac{16}{5}\right\} \subseteq \mathbb{R}^3$, and is of dimension 0.

(b) $\left\{x=\frac{523}{60}, y=-\frac{317}{60}, z=-\frac{13}{12}\right\} \subseteq \mathbb{R}^3$, and is of dimension 0.

(c) $\left\{w=1-\frac{1}{2}y+\frac{1}{2}z, x=1-y-z, y=y, z=z\right\} \subseteq \mathbb{R}^4$, and is of dimension 2.

(d) $\left\{w=-\frac{3}{2}, x=\frac{111}{50}, y=\frac{52}{25}, z=\frac{1}{5}\right\} \subseteq \mathbb{R}^4$, and is of dimension 0.

(e) $\left\{x=-\frac{1}{5}-\frac{1}{5}z, y=\frac{2}{5}+\frac{19}{10}z, z=z\right\} \subseteq \mathbb{R}^3$, and is of dimension 1.

(f) $\left\{x=-\frac{11}{56}, y=\frac{41}{112}, z=-\frac{1}{56}\right\} \subseteq \mathbb{R}^3$, and is of dimension 0.

5. Write out the solutions to each system from problem 3 in column matrix format using scalar multiplication by the independent variables.

(a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{203}{55} \\ -\frac{119}{55} \\ \frac{16}{5} \end{bmatrix}$ (b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{523}{60} \\ -\frac{317}{60} \\ -\frac{13}{12} \end{bmatrix}$

(c) $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} \frac{1}{2} \\ -1 \\ 0 \\ 1 \end{bmatrix} z$ (d) $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ \frac{111}{50} \\ \frac{52}{25} \\ \frac{1}{5} \end{bmatrix}$

(e) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} \\ \frac{2}{5} \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{5} \\ \frac{19}{10} \\ 1 \end{bmatrix} z$ f) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{11}{56} \\ \frac{41}{112} \\ -\frac{1}{56} \end{bmatrix}$

6. Would Gauss-Jordan elimination be easier to implement in any of the systems of problem 3 if you were allowed to use the row swap operation? If so, explain and go through the process of Gauss-Jordan elimination again, using the row swap operation.

The matrices in parts (a), (d) and (f) could have utilized the row swapping operation in the Gauss-Jordan elimination process.

(a)
- step 1) 2row1 + row2 $\to$ row2 step 2) -2row1 + row3 $\to$ row3
- step 3) swap row2 and row3 step 4) $\frac{1}{11}$row2 $\to$ row 2
- step 5) 3row2 + row1 $\to$ row1 step 6) $\frac{1}{5}$row3 $\to$ row3
- step 7) $-\frac{4}{11}$row3 +row2$\to$ row2 step 8) $-\frac{23}{11}$row3 +row1$\to$ row1

(d)
- step 1) -2row1 + row2 $\to$ row2 step 2) row1 + row3 $\to$ row3
- step 3) -4row1 + row4 $\to$ row4 step 4) swap row2 and row4
- step 5) $\frac{1}{2}$row2 $\to$ row2 step 6) -2row2 + row3 $\to$ row3
- step 7) -row2 + row1 $\to$ row1 step 8) $\frac{1}{5}$row3 $\to$ row3
- step 9) $\frac{3}{2}$row3 + row2 $\to$ row2 step 10) $-\frac{5}{2}$row3 + row1 $\to$ row1
- step 11) $-\frac{1}{5}$row4 $\to$ row4 step 12) $-\frac{13}{5}$row4 + row3 $\to$ row3
- step 13) $\frac{8}{5}$row4 +row2 $\to$ row2

(f)
- step 1) $\frac{1}{3}$row1 $\to$ row1 step 2) 2row1 +row2 $\to$ row2
- step 3) -5row1+row3 $\to$ row3 step 4) 3row1 + row4 $\to$ row4
- step 5) $\frac{3}{20}$row2 $\to$ row2 step 6) -8row2 + row4 $\to$ row4
- step 7) $\frac{20}{3}$row2 + row3 $\to$ row3 step 8) $-\frac{4}{3}$row2 + row1 $\to$ row1
- step 9) swap row3 and row4 step 10) $\frac{5}{56}$row3 $\to$ row3
- step 11) $\frac{19}{10}$row3 + row2 $\to$ row2 step 12) $-\frac{1}{5}$row3 + row1 $\to$ row1

7. Perform Gauss-Jordan elimination on the following matrices.

(a) $\begin{bmatrix} 2 & -3 & 6 \\ 8 & -4 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -\frac{9}{8} \\ 0 & 1 & -\frac{11}{4} \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -3 & 6 & 1 \\ 8 & -4 & 2 & 3 \\ 7 & -3 & 2 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & \frac{3}{2} \end{bmatrix}$

(c) $\begin{bmatrix} 2 & -3 & 4 & -5 & -10 \\ 1 & 1 & -1 & 1 & 4 \\ 3 & 5 & -9 & 7 & 24 \\ 2 & 2 & -2 & 3 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

8. Given the following two systems of equations:

(a) $\begin{aligned} 3x + 4y &= -1 \\ 4x - 2y &= 6 \end{aligned}$ (b) $\begin{aligned} 3x + 4y &= 9 \\ 4x - 2y &= -10 \end{aligned}$

Explain how the following matrix can be used to solve both systems simultaneously, then do so:

$$\left[\begin{array}{cc|cc} 3 & 4 & -1 & 9 \\ 4 & -2 & 6 & -10 \end{array}\right]$$

To solve system (a), we would *rref* the augmented matrix

$$A = \left[\begin{array}{cc|c} 3 & 4 & -1 \\ 4 & -2 & 6 \end{array}\right].$$

In a similar fashion, system (b) can be solved by row reducing the augmented matrix

$$B = \left[\begin{array}{cc|c} 3 & 4 & 9 \\ 4 & -2 & -10 \end{array}\right].$$

These can be done independently, however, since the left-hand sides of both augmented matrices are the same (hence they only differ in the last column), any row operations applied to $A$ to reduce $A$ to *rref* form will be the same row operations that are needed to reduce $B$ to *rref* form. Since each column is independent of every other column when row reducing, we can simply augment the last column of $B$ onto the end of $A$, row reduce, and recover the solution to both systems at once!

Notice that

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \quad \text{rref}(B) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix},$$

while

$$\left[\begin{array}{cc|cc} 3 & 4 & -1 & 9 \\ 4 & -2 & 6 & -10 \end{array}\right] \longrightarrow \left[\begin{array}{cccc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 3 \end{array}\right].$$

From the *rref* matrices above, it should be clear that our argument holds up to example. Furthermore, the solutions to system (a) and system (b) are the third and fourth columns of the twice augmented matrix, respectively. We can write these solutions explicitly as $\{x = 1, y = -1\}$ and $\{x = -1, y = 3\}$.

9. Is it possible for a *rref* matrix to have more than one row whose corresponding linear equation is $0 = 1$? If no, explain why not. If yes, then give an example.

Clearly a matrix, corresponding to a linear system, can have multiple rows having the form of the equation $0 = 1$. However, the claim is that such a matrix would not be in *rref* form. To see this, if rows $n$ and $m$ of a *rref* matrix have the form $0 = 1$, then both rows $n$ and $m$ must consist entirely of all zeros, except for the entry in the last column of each, both of which must be nonzero. If the value in the last column of row $n$ is $a$, and of row $m$ is $b$, then multiplying row $n$ by $-\frac{b}{a}$ and adding it to row $m$ will zero out all of row $m$, contradicting the fact that the matrix was in *rref* form. Therefore, we can conclude that a *rref* matrix can have at most one row whose corresponding linear equation is $0 = 1$.

10. A square matrix is called upper triangular if all of its entries below the main diagonal from upper left to lower right are 0. If $A$ is a square matrix, then must *rref*$(A)$ be upper triangular? If yes, explain why. If no, then give an example.

By the definition of the process of performing Gauss-Jordan elimination, we start in the upper left corner and zero all elements in the first column below the first entry. Once complete, we move to the entry in row 2 column 2. The entries in column 2 below row 2 are then zeroed out, and then the one in row 1 column 2. The process, moving down the diagonal continues until either 1) you end up in the last row, or 2) you have a 0 in the diagonal entry, in which case you shift to the right one element in the matrix, and continue on. In either case, all values below the diagonal will be zero. Therefore, if we start with an upper triangular matrix, its *rref* form will be upper triangular.

11. What would have to be true about a linear system so that the *rref* matrix of the augmented matrix of this system has all rows of all zeroes except for the first row? Give examples of the possibilities.

For this scenario to occur, we would have to have each equation be a scalar multiple of the first, and only first, equation.

For example, if we start with the simple equation $x + 2y = 3$, and take several scalar multiples of this equation, such as $-2x - 4y = -6$ and $\frac{1}{2}x + y = \frac{3}{2}$, and row reduce the corresponding augmented matrices, we will be reduced to just one equation (in this example, we will get exactly the first equation since the coefficient in front of the $x$ variable is 1).

$$\begin{array}{ccc} \text{matrix} & & \textit{rref} \text{ matrix} \\ \begin{bmatrix} 1 & 2 & 3 \\ -2 & -4 & -6 \end{bmatrix} & \longrightarrow & \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 2 & 3 \\ \frac{1}{2} & 1 & \frac{3}{2} \end{bmatrix} & \longrightarrow & \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 2 & 3 \\ \frac{1}{2} & 1 & \frac{3}{2} \\ -2 & -4 & -6 \end{bmatrix} & \longrightarrow & \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}. \end{array}$$

12. What would have to be true about a linear system so that the *rref* matrix of the augmented matrix of this system is an identity matrix? Give examples of the possibilities.

First off, to end up with the $n \times n$ identity matrix we must have $n$ equations with $n-1$ variables. This implies that we have an overdetermined system. Therefore, we must have an overdetermined system, with the added requirement that no equation can be expressed as a linear combination of other rows, so that no row of all zeroes appears in the resulting *rref* matrix. Of course, if one row reduces an augmented system and the end result is the $n \times n$ identity matrix, the system has no solution.

As an example consider the system of equations

$$\begin{aligned} x + 2y &= 1 \\ -x + 3y &= 1 \\ x - 2y &= 2, \end{aligned}$$

whose augmented matrix is given by:

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 3 & 1 \\ 1 & -2 & 2 \end{bmatrix}.$$

Notice that this is a system of three equations in the two variables $x$ and $y$, hence this system is overdetermined. When we row reduce, either we should get a row of all zeros, or we will end up with the $3 \times 3$ identity matrix. In this example, row reducing does result in $I_3$.

13. What would have to be true about a linear system so that the *rref* matrix of the augmented matrix of this system is an identity matrix to the left of the last column? Give examples of the possibilities.

For a linear system to be row reduced as described above, we must have a system of $n$ equations in $n$ variables, and a solution of dimension 0, i.e. a single solution. If the $n$ variables are $x_1, x_2, \ldots, x_n$, and the entries in the last column are $a_1, a_2, \ldots, a_n$, then the matrix

$$\begin{bmatrix} 1 & \ldots & 0 & a_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 1 & a_n \end{bmatrix}$$

corresponds to the unique solution $x_k = a_k$, for $1 \leq k \leq n$.

## 3.2 Elementary Matrices

1. Given a system of $n$ equations in $n$ variables, what would the maximum number of left multiplications by elementary matrices be to convert the original augmented matrix representation of the system to *rref* form?

Since the augmented matrix will be $n \times n + 1$, and we need to convert the everything but the last column to the $n \times n$ identity matrix, it should take $n^2$ elementary row operations. If a row swap must be preformed, then that means that another row has a zero in the correct place, thus negating the requirement of adding any more left multiplications to the process. The final answer is $n^2$.

2. The example used in this section was a three equation, three variable system. Can elementary matrices be used on nonsquare systems? What are the restrictions?

We are restricted to elementary matrices with square dimension equal to the number of equations in the non-square system. If there are $n$ equations, multiplication by elementary matrices will only work on the first $n$ variables in the system, resulting in a matrix which may not be in simplest *rref* form.

3. Use left multiplication by elementary matrices to reduce the following systems of equations as far as possible. Also determine if the resulting matrix is in reduced *rref* form.

(a) $\begin{array}{r} 2x - 3y = 7 \\ -2x + 5y = 1 \end{array}$ in augmented form is $\begin{bmatrix} 2 & -3 & 7 \\ -2 & 5 & 1 \end{bmatrix}$

$$E_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, E_4 = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix}$$

$$E_4E_3E_2E_1A = \begin{bmatrix} 1 & 0 & \frac{19}{2} \\ 0 & 1 & 4 \end{bmatrix}$$

(b) $\begin{array}{r} -7x + 2y = 5 \\ 6x + 3y = 4 \end{array}$ in augmented form is $\begin{bmatrix} -7 & 2 & 5 \\ 6 & 3 & 4 \end{bmatrix}$

$$E_1 = \begin{bmatrix} -\frac{1}{7} & 0 \\ 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \\ -6 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{7}{33} \end{bmatrix}, E_4 = \begin{bmatrix} 1 & \frac{2}{7} \\ 0 & 1 \end{bmatrix}$$

$$E_4E_3E_2E_1A = \begin{bmatrix} 1 & 0 & -\frac{7}{33} \\ 0 & 1 & \frac{58}{33} \end{bmatrix}$$

(c) $\begin{array}{r} x - 3y + z = 6 \\ -2x + 6y + 3z = 4 \\ 2x + 5y + 6z = 1 \end{array}$ in augmented form is $\begin{bmatrix} 1 & -3 & 1 & 6 \\ -2 & 6 & 3 & 4 \\ 2 & 5 & 6 & 1 \end{bmatrix}$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{11} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix},$$

$$E_7 = \begin{bmatrix} 1 & 0 & -\frac{23}{11} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{4}{11} \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_8E_7E_6E_5E_4E_3E_2E_1A = \begin{bmatrix} 1 & 0 & 0 & -\frac{203}{55} \\ 0 & 1 & 0 & -\frac{119}{55} \\ 0 & 0 & 1 & \frac{16}{5} \end{bmatrix}$$

(d) $\begin{array}{l} 2x+3y-5z=7 \\ 3x+2y+7z=8 \\ 4x+6y+2z=1 \end{array}$ in augmented form is $\begin{bmatrix} 2 & 3 & -5 & 7 \\ 3 & 2 & 7 & 8 \\ 4 & 6 & 2 & 1 \end{bmatrix}$

$$E_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{2}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & -\frac{3}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{12} \end{bmatrix},$$

$$E_7 = \begin{bmatrix} 1 & 0 & -\frac{31}{5} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{29}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_8E_7E_6E_5E_4E_3E_2E_1A = \begin{bmatrix} 1 & 0 & 0 & \frac{523}{60} \\ 0 & 1 & 0 & -\frac{317}{60} \\ 0 & 0 & 1 & -\frac{13}{12} \end{bmatrix}$$

(e) $\begin{array}{l} -3y+z=6 \\ -2x+3z=4 \\ 2x+5y=1 \end{array}$ in augmented form is $\begin{bmatrix} 0 & -3 & 1 & 6 \\ -2 & 0 & 3 & 4 \\ 2 & 5 & 0 & 1 \end{bmatrix}$

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{3}{14} \end{bmatrix},$$

$$E_7 = \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_8E_7E_6E_5E_4E_3E_2E_1A = \begin{bmatrix} 1 & 0 & 0 & \frac{79}{28} \\ 0 & 1 & 0 & -\frac{13}{14} \\ 0 & 0 & 1 & \frac{45}{14} \end{bmatrix}$$

(f) $\begin{array}{c} w - x - 3y + z = 6 \\ w + 2x - y + 3z = 4 \\ -w + 2x + 4z = 1 \\ x - 2y + 5z = 1 \end{array}$ in augmented form is $\begin{bmatrix} 1 & -1 & -3 & 1 & 6 \\ 1 & 2 & -1 & 3 & 4 \\ -1 & 2 & 0 & 4 & 1 \\ 0 & 1 & -2 & 5 & 1 \end{bmatrix}$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix},$$

$$E_7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{3}{11} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_8 = \begin{bmatrix} 1 & 0 & \frac{7}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_9 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$E_{10} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{8}{3} & 1 \end{bmatrix}, \quad E_{11} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{11}{13} \end{bmatrix}, E_{12} = \begin{bmatrix} 1 & 0 & 0 & \frac{12}{11} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$E_{13} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{16}{11} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E_{14} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{13}{11} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$E_{14}E_{13}E_{12}E_{11}E_{10}E_9E_8E_7E_6E_5E_4E_3E_2E_1A = \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{41}{13} \\ 0 & 1 & 0 & 0 & \frac{72}{13} \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -\frac{43}{13} \end{bmatrix}$$

4. Use left multiplication by elementary matrices to reduce the following systems of equations to *rref* form (or as close as possible). You may leave your

answer in *rref* matrix form.

(a) $\begin{array}{r} x - 3y + z = 6 \\ -2x + 6y + 3z = 4 \end{array}$ in augmented form is $\begin{bmatrix} 1 & -3 & 1 & 6 \\ -2 & 6 & 3 & 4 \end{bmatrix}$

$$E_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$E_1C = \begin{bmatrix} 1 & -3 & 1 & 6 \\ 0 & 0 & 5 & 16 \end{bmatrix}$$

(b) $\begin{array}{r} 2x + 3y = 7 \\ 3x + 2y = 8 \\ -2x - 5y = -2 \end{array}$ in augmented form is $\begin{bmatrix} 2 & 3 & 7 \\ 3 & 2 & 8 \\ -2 & -5 & -2 \end{bmatrix}$

$$E_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{2}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & -\frac{3}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix},$$

$$E_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{7} \end{bmatrix}, \quad E_8 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_9 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_9E_8E_7E_6E_5E_4E_3E_2E_1C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) $\begin{array}{r} w + x + z = 1 \\ x + 3z = 4 \\ 6y + 2z = 1 \end{array}$ in augmented form is $\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 6 & 2 & 1 \end{bmatrix}$

$$E_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix},$$

$$E_2E_1C = \begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

(d) $\begin{array}{r} 2x+3y=7 \\ 3x+2y=8 \\ x-y=1 \end{array}$ in augmented form is $\begin{bmatrix} 2 & 3 & 7 \\ 3 & 2 & 8 \\ 1 & -1 & 1 \end{bmatrix}$

$$E_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{2}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{5}{2} & 1 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & -\frac{3}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_6E_5E_4E_3E_2E_1C = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

5. Use the elementary matrices from part (a) of problem 3, on the following two corresponding systems. Explain what this implies.

(a) $\begin{array}{r} 2x-3y=3 \\ -2x+5y=-4 \end{array}$ in augmented form is $\begin{bmatrix} 2 & -3 & 3 \\ -2 & 5 & -4 \end{bmatrix}$

$$E_4E_3E_2E_1A = \begin{bmatrix} 1 & 0 & \frac{27}{4} \\ 0 & 1 & \frac{7}{2} \end{bmatrix}$$

(b) $\begin{array}{r} 2x-3y=-2 \\ -2x+5y=-1 \end{array}$ in augmented form is $\begin{bmatrix} 2 & -3 & -2 \\ -2 & 5 & -1 \end{bmatrix}$

$$E_4E_3E_2E_1A = \begin{bmatrix} 1 & 0 & -\frac{13}{4} \\ 0 & 1 & -\frac{3}{2} \end{bmatrix}$$

What this implies, is that as long as the left hand sides of the equations are the same, the elementary matrices do not change. Thus, one can solve a multitude of problems, where only the right hand side changes, using the same sequence of elementary matrix multiplications.

6. (a) Give examples, $E_1$, $E_2$, and $E_3$, of each of the 3 types of $4\times4$ elementary matrices.

Obviously answers will vary, but here is one set of examples:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

(b) Next, find for each of the elementary matrices $E_1$, $E_2$, and $E_3$, of part a another elementary matrix $F_1$, $F_2$, and $F_3$ of the same type and size as the corresponding $E$ so that $F_k E_k = I_4$ and $E_k F_k = I_4$ for $k = 1, 2, 3$. Each $F_k = E_k^{-1}$, that is, each $F_k$ is $E_k$'s multiplicative inverse and vice versa.

$$F_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, F_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, F_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

(c) Compute $rref(E_k|\, I_4)$, for $k = 1, 2, 3$ where $(E_k|\, I_4)$ is the $4 \times 8$ augmented matrix. What is the result of these 3 $rref$'s?

$$rref\,(E_1|\, I_4) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$rref\,(E_2|\, I_4) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$rref\,(E_3|\, I_4) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

The right half of each $rref\,(E_k|\, I_4)$ corresponds to the $F_k$'s, for $k = 1, 2, 3$.

7. (a) Give examples $E_1, E_2, E_3$, of each of the 3 types of $4 \times 4$ elementary matrices.

$$E_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Next, compute $E_1^2$, $E_2^2$, $E_3^2$ and $E_1^3$, $E_2^3$, $E_3^3$. Now give a general formula for $E_1^m$, $E_2^m$, $E_3^m$, where $m$ is any positive integer.

$$E_1^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E_2^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E_3^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 6 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_1^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, E_2^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -27 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E_3^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 9 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_1^{2n} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad E_1^{2n+1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$E_2^m = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (-3)^m & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E_3^m = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3m & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) If you did problem 6, then find a general formula for $E_1^m$, $E_2^m$, $E_3^m$, where $m$ is any negative integer.

$$E_1^{2n} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_1^{2n+1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$E_2^m = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2^m & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E_3^m = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -m & 0 & 1 \end{bmatrix}$$

(d) Can you put parts (b) and (c) of this problem together to get a general formula for $E_1^m$, $E_2^m$, $E_3^m$, where $m$ is any integer?

For type I matrices, of $E_1^{2m+1} = E_1$ and $E_1^{2m} = I_n$.

For type II matrices, you simply raise the entry on the diagonal not equal to one to the $m^{th}$ power.

For type III matrices, you simply multiply the nonzero entre off the diagonal by $m$.

8. Let $E$ and $F$ be two $3 \times 3$ elementary matrices. What must be true about $E$ and $F$ so that $EF = FE$, that is, $E$ and $F$ commute?

If $E$ and $F$ are any two type II elementary matrices, then they will commute. Furthermore, type III elementary matrices with entries in the same column also commute, as discussed in the text. However, they must be in the same column. If $E$ and $F$ are inverses of each other, as described in problem 6b), they will also commute.

9. (a) Which type of elementary matrix is always a diagonal matrix? A diagonal matrix is a square matrix $A$, where all entries $A_{i,j} = 0$, when $i \neq j$.

Elementary matrices of type II are diagonal.

(b) Which type of elementary matrix is always an upper or lower triangular matrix? An upper or lower triangular matrix is a square matrix $A$ where in the upper triangular case all entries $A_{i,j} = 0$ when $i > j$ while in the lower triangular case all entries $A_{i,j} = 0$ when $i < j$.

Elementary matrices of type III will always be upper or lower triangular, since they are constructed as the identity matrix with one non-zero off diagonal entry.

It should be noted that type I elementary matrices are not diagonal nor triangular.

## 3.3 Sensitivity of Solutions to Error in the Linear System

Consider the following systems of equations:

(a) $-x + 0.048y = 6$
$2x - 0.1y = 24$

(b) $-x + 0.049y = 6$
$2x - 0.1y = 24$

(c) $-x + 0.048y = 6.1$
$2x - 0.1y = 24$

(d) $-x + 0.048y = 6$
$2.01x - 0.1y = 24$

We will denote system (a) as the exact system, while (b), (c), and (d) will be approximate systems. Answer the following questions:

1. Solve all four systems for the variables $x$ and $y$.

(a) $\{x = -438, y = -9000\}$

(b) $\{x = -888, y = -18000\}$

(c) $\{x = -440.5, y = -9050\}$

(d) $\{x = -497.7272727, y = -10244.31818\}$

2. Compute the Euclidean distance between the solution to the actual system and each of the three approximate systems.

(a) $d((x_a, y_a), (x_b, y_b)) = 8.1202500 \times 10^7$

(b) $d((x_a, y_a), (x_c, y_c)) = 2506.25$

(c) $d((x_a, y_a), (x_d, y_d)) = 1.551895080 \times 10^6$

3. The *Frobenius norm* of an $m \times n$ matrix $A$ (with potentially complex entries) is one way to measure the magnitude, or length, of a matrix. It is given by the following formula:

$$\parallel A \parallel_F = \sqrt{\sum_{i=i}^{m} \sum_{j=1}^{n} |A_{i,j}|^2}$$

The Frobenius distance between two matrices $A$ and $B$ of the same size is $\parallel A - B \parallel_F$. Use this definition of the Frobenius norm to compute the distance between the augmented matrix corresponding to the exact system and the augmented matrices corresponding to approximate systems.

(a) $\parallel A - B \parallel_F = 0.001$

(b) $\parallel A - C \parallel_F = 0.1$

(c) $\parallel A - D \parallel_F = 0.01$

4. One might expect that the corresponding Frobenius norms from problem 3 should be arranged in the same order as the distances found in problem 2. Can you come up with a reason as to why this is not the case?

Solving each equation $Ax + By = C$ for $y$ puts each into the standard $y = mx + b$ form. If there is a large relative difference in the values of $B$ for the exact and approximate system (for instance if $B$ is small), this will greatly affect the values of $m$ and $b$. However, if $A$ is slightly changed, or $C$, all that happens will be slight changes in slopes or shifting of the lines up and down. Of course, everything depends on the relative values of $A$, $B$ and $C$.

Consider the following systems of equations:

(a)
$$\begin{aligned} 6w - 3x + 2y &= 7 \\ w - 2x + 4y &= 5 \\ 4w + x - 7y &= 0 \end{aligned}$$

(b)
$$\begin{aligned} 6w - 3x + 2y &= 7 \\ w - 2x + 4y &= 5 \\ 4w + x - 6.1y &= 0 \end{aligned}$$

(c)
$$\begin{aligned} 6w - 3x + 2y &= 7 \\ w - 2x + 4y &= 5 \\ 5w + x - 7y &= 0 \end{aligned}$$

(d)
$$\begin{aligned} 6w - 3x + 2y &= 7 \\ w - 2x + 4y &= 5 \\ 4w + x - 7y &= -2 \end{aligned}$$

As before, we will denote system (a) as the exact system, while (b), (c), and (d) will be approximate systems. Answer the following questions:

5. Solve all four systems for the variables $w$, $x$, and $y$.

(a) $\{w = -2.777777778, x = -9.888888889, y = -3\}$

(b) $\{w = -26.77777776, x = -75.88888885, y = -29.99999998\}$

(c) $\{w = -25, x = -71, y = -28\}$

(d) $\{w = -1, x = -5, y = -1\}$

6. Compute the Euclidean distance between the solution to the exact system and each of the three approximate systems.

(a) $d((w_a, x_a, y_a), (w_b, x_b, y_b)) = 75.23961718$

(b) $d((w_a, x_a, y_a), (w_c, x_c, y_c)) = 69.66631224$

(c) $d((w_a, x_a, y_a), (w_d, x_d, y_d)) = 5.573304980$

7. Compute the Frobenius norm of the distance between the augmented matrix corresponding to the exact system and the augmented matrices corresponding to approximate systems.

(a) $\| A - B \|_F = 0.09$

(b) $\| A - C \|_F = 1$

(c) $\| A - D \|_F = 2$

# Chapter 4

# Applications of Linear Systems and Matrices

## 4.1 Applications of Linear Systems to Geometry

1. Find equations of the circles that pass through the following sets of points:

We will use the equation $Dx + Ey + F = -(x^2 + y^2)$ for this problem.

(a) $\left\{\left(0, -1-\sqrt{3}\right), (1,1), \left(1+\sqrt{3}, -2\right)\right\}$

Our system of three equations and three unknowns is given by

$$\begin{aligned} \left(-1-\sqrt{3}\right) E + F &= -\left(-1-\sqrt{3}\right)^2 \\ D + E + F &= -2 \\ \left(1+\sqrt{3}\right) D - 2\,E + F &= -\left(1+\sqrt{3}\right)^2 - 4. \end{aligned}$$

Solving this system of equations gives the solution $\{D = -2, E = 2, F = -2\}$. Therefore, the circle passing through these three points is

$$x^2 + y^2 - 2x + 2y - 2 = 0.$$

(b) $\{(3,7), (3,1), (6,4)\}$

Our system of three equations and three unknowns is given by

$$\begin{aligned} 3\,D + 7\,E + F &= -58 \\ 3\,D + E + F &= -10 \\ 6\,D + 4\,E + F &= -52. \end{aligned}$$

Solving this system of equations gives the solution $\{D = -6, E = -8, F = 16\}$. Therefore, the circle passing through these three points is

$$x^2 + y^2 - 6x - 8y + 16 = 0.$$

2. Find equations of the planes that pass through the following sets of points:

We will attempt to use the form, $E\,x + F\,y + G\,z = 1$, of a plane for the points given. Here we assumed the constant term was nonzero and divided through by it to reduce our equation to one with only three unknowns.

(a) $\{(1, -2, -5), (-3, -2, 1), (-3, -1, 3)\}$

Our system of three equations in terms of $E$, $F$ and $G$ are given by:

$$\begin{aligned} E - 2\,F - 5\,G &= 1 \\ -3\,E - 2\,F + G &= 1 \\ -3\,E - F + 3\,G &= 1. \end{aligned}$$

Solving this system of equations gives the solution $\{E = 3, F = -4, G = 2\}$. Therefore, the plane passing through these three points is

$$3\,x - 4\,y + 2\,z = 1.$$

(b) $\{(-4, 3, 8), (-6, -2, 1), (-3, 0, 3)\}$

Our system of three equations in terms of $E$, $F$ and $G$ are given by:

$$\begin{aligned} -4\,E + 3\,F + 8\,G &= 1 \\ -6\,E - 2\,F + G &= 1 \\ -3\,E + 3\,G &= 1. \end{aligned}$$

Solving this system of equations gives the solution $\left\{E = \frac{4}{21}, F = -\frac{17}{21}, G = \frac{11}{21}\right\}$. Therefore, the plane passing through these three points is

$$\frac{4}{21}\,x - \frac{17}{21}\,y + \frac{11}{21}\,z = 1.$$

3. Find the equation of the plane that passes through the following points:

$$\{(1, -2, 11), (-1, 1, -7), (2, 1, 2)\}$$

We will attempt to use the form, $E\,x + F\,y + G\,z = 1$, of a plane for the points given. Here we assumed the constant term was nonzero and divided through by it to reduce our equation to one with only three unknowns.

Our system of three equations in terms of $E$, $F$ and $G$ are given by:

$$\begin{aligned} E - 2\,F + 11\,G &= 1 \\ -E + F - 7\,G &= 1 \\ 2\,E + F + 2\,G &= 1. \end{aligned}$$

In matrix form, we have

$$\begin{bmatrix} 1 & -2 & 11 & 1 \\ -1 & 1 & -7 & 1 \\ 2 & 1 & 2 & 1 \end{bmatrix},$$

which row reduced gives

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The last row yields the equation $0 = 1$, hence the system has no solution. What does this mean? Well, we cannot fit the three points given to a plane of the form $E\,x + F\,y + G\,z = 1$. Note that we assumed the constant term was nonzero, hence divided through by it. This must be false, hence the constant term must be zero. Let us try using a different form of the plane, namely: $E\,x + F\,y + G\,z = 0$ Now we really appear to have an extra variable, since we could divide through by any of $E$, $f$ or $G$, however we cannot be sure that any of them are nonzero (although this can be verified by looking at the points in question).

Our new system of three equations in terms of $E$, $F$ and $G$ are given by:

$$\begin{aligned} E - 2\,F + 11\,G &= 0 \\ -E + F - 7\,G &= 0 \\ 2\,E + F + 2\,G &= 0. \end{aligned}$$

Solving this system of equations gives the solution $\{E = -3\,G, F = 4\,G\}$. Notice that $G$ is an independent variable, hence we can set it to an nonzero value we wish. In this case, let $G = 1$, which gives $E = -3$ and $F = 4$. The equation of our plane is thus given by

$$-3\,x + 4\,y + z = 0.$$

4. A sphere of radius $r$, centered at the point $(a, b, c)$, can be expressed by the equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

Construct a linear system of equations that can be solved to find the sphere that fits a set of data points in $\mathbb{R}^3$. How many points are required to determine a unique sphere?

If we expand the above expression, we get

$$x^2 - 2ax + a^2 + y^2 - 2ab + b^2 + z^2 - 2cz + c^2 = r^2.$$

Upon setting $A = -2a$, $B = -2b$, $C = -2c$ and $D = a^2 + b^2 + c^2 - r^2$ we arrive at the new equation and moving all terms with no constants to the right hand side yields

$$Ax + By + Cz + D = -(x^2 + y^2 + z^2).$$

We have 4 unknowns $A$, $B$, $C$ and $D$, from which we can easily compute $a$, $b$, $c$ and $d$. Therefore, we need 4 points to determine a unique sphere.

5. Find the equation of the spheres that fit the following set of points:

(a) $\left\{\left(2, 3, 3 - 2\sqrt{3}\right), \left(4, 1 + 2\sqrt{3}, 3\right), (2, 1, -1), (6, 1, 3)\right\}$

Using the equation from the answer to problem 3 gives the following system of equations:

$$\begin{aligned} 2\,A + 3\,B + C\left(3 - 2\,\sqrt{3}\right) + D &= -13 - \left(3 - 2\,\sqrt{3}\right)^2 \\ 4\,A + B\left(1 + 2\,\sqrt{3}\right) + 3\,C + D &= -25 - \left(1 + 2\,\sqrt{3}\right)^2 \\ 2\,A + B - C + D &= -6 \\ 6\,A + B + 3\,C + D &= -46. \end{aligned}$$

Solving for $A$, $B$, $C$ and $D$ gives $\{A = -4, B = -2, C = -6, D = -2\}$. In terms of $a$, $b$, $c$ and $r$ we get $\{a = 2, b = 1, c = 3, r = 4\}$. So the equation of the sphere is

$$(x - 2)^2 + (y - 1)^2 + (z - 3)^2 = 4^2.$$

(b) $\{(-3, -1, -3), (0, -4, 1), (-2, 0, 1), (1, -1, 1)\}$

Using the given four points, we end up with the following systems of equations:

$$\begin{aligned} -3\,A - B - 3\,C + D &= -19 \\ -4\,B + C + D &= -17 \\ -2\,A + C + D &= -5 \\ A - B + C + D &= -3. \end{aligned}$$

Solving for $A$, $B$, $C$ and $D$ gives $\{A = 2, B = 4, C = 2, D = -3\}$. In terms of $a$, $b$, $c$ and $r$ we get $\{a = -1, b = -2, c = -1, r = 3\}$. So the equation of the sphere is

$$(x + 1)^2 + (y + 2)^2 + (z + 1)^2 = 3^2.$$

6. Explain what happens when you attempt to find the equation of a conic for which three of the given points are collinear.

Given five distinct points, three of which are collinear, then the resulting conic is called *degenerate*. In this situation, you will either end up with two distinct lines, or one double line (i.e. a line of the form $(ax+by+c)^2=0$). As an example, consider the following sets of points:

$$\{(-1,1),(0,1),(1,1),(2,3),(-2,0)\},$$

which results in the following system of equations:

$$\begin{aligned} A-B+C-D+E+F&=0\\ C+E+F&=0\\ A+B+C+D+E+F&=0\\ 4\,A+6\,B+9\,C+2\,D+3\,E+F&=0\\ 4\,A-2\,D+F&=0. \end{aligned}$$

The solution to this system of equations is

$$\left\{A=0,C=-\frac{4}{3}\,B,D=-B,E=\frac{10}{3}\,B,F=-2\,B\right\},$$

and setting $B=1$ gives the conic

$$xy-\frac{4}{3}\,y^2-x+\frac{10}{3}\,y-2=0.$$

Notice that this factors as the product of two lines, as depicted below:

$$\frac{1}{3}\,(y-1)\,(-4\,y+6+3\,x)=0,$$

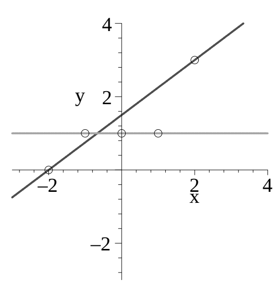


Figure 4.1: Degenerate conic section in this example is the product of two lines.

7. (See Section 4.4 for more details about rotations in the plane.) Let $(x', y')$ be a new coordinate system that is a rotation about the origin through the angle $\theta$ of the standard $(x, y)$ Cartesian coordinate system. (This is actually a rotation about the origin of the $x$ and $y$ axes to produce the new $x'$ and $y'$ axes, respectively.) Then these two coordinate systems are related by the equations

$$\begin{aligned} x' &= \cos(\theta)\,x - \sin(\theta)\,y \\ y' &= \sin(\theta)\,x + \cos(\theta)\,y \end{aligned}$$

or the single matrix equation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Let

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

be a conic section in the $xy$-coordinate system. Find the equation of this conic section

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$$

in the $x'y'$-coordinate system. In particular, find formulas for the coefficients $A'$ through $F'$ in terms of $A$ through $F$ and $\theta$. Also, show that the discriminants of both equations are equal, that is,

$$B'^2 - 4A'C' = B^2 - 4AC$$

Also, find a formula in terms of $A$ through $F$ for the angle $\theta$ which makes $B' = 0$. Now find this angle $\theta$ and corresponding values of $A'$ through $F'$ for the conic given by

$$4x^2 + 6xy - 2y^2 + 7x + y - 1 = 0.$$

First, in order to write the the conic in terms of the rotated variables $x'$ and $y'$, we need to write $x$ and $y$ in terms of $x'$ and $y'$. To do this, we simply compute the inverse rotation matrix (which is equivalent to replacing $\theta$ with $-\theta$ in this situation):

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

Therefore, we have $x = \cos(\theta)\,x' + \sin(\theta)\,y'$ and $y = -\sin(\theta)\,x' + \cos(\theta)\,y'$, which we substitute into the conic equation:

$$\begin{aligned} &A\left(\cos(\theta)\,x' + \sin(\theta)\,y'\right)^2 + B\left(\cos(\theta)\,x' + \sin(\theta)\,y'\right)\left(-\sin(\theta)\,x' + \cos(\theta)\,y'\right) \\ &+ C\left(-\sin(\theta)\,x' + \cos(\theta)\,y'\right)^2 + D\left(\cos(\theta)\,x' + \sin(\theta)\,y'\right) \\ &+ E\left(-\sin(\theta)\,x' + \cos(\theta)\,y'\right) + F = 0. \end{aligned}$$

Upon multiplying everything out and collecting terms, we end up with the following:

$$\left(A\cos^2(\theta) - B\cos(\theta)\sin(\theta) + C\sin^2(\theta)\right)x'^{\,2} + \left(2A\cos(\theta)\sin(\theta) + B\cos^2(\theta)\right.$$
$$\left.-B\sin^2(\theta) - 2C\cos(\theta)\sin(\theta)\right)x'y' + \left(A\sin^2(\theta) + B\sin(\theta)\cos(\theta)\right.$$
$$\left.+C\cos^2(\theta)\right)y'^{\,2} + (D\cos(\theta) - E\sin(\theta))\,x' + (D\sin(\theta) + E\cos(\theta))\,y' + F = 0$$

Thus, we get from the above expansion that

$$\begin{aligned}
A' &= A\cos^2(\theta) - B\cos(\theta)\sin(\theta) + C\sin^2(\theta)\\
B' &= 2A\cos(\theta)\sin(\theta) + B\cos^2(\theta) - B\sin^2(\theta) - 2C\cos(\theta)\sin(\theta)\\
C' &= A\sin^2(\theta) + B\sin(\theta)\cos(\theta) + C\cos^2(\theta)\\
D' &= D\cos(\theta) - E\sin(\theta)\\
E' &= D\sin(\theta) + E\cos(\theta)\\
F' &= F.
\end{aligned}$$

To show that the two discriminants are equal, we simply substitute into $B'^{\,2} - 4A'C'$ the values above, expand, perform simplifications with the end result being $B^2 - 4AC$.

Next, setting $B' = 0$ results in the equation

$$2\cos(\theta)\sin(\theta)(C - A) = B(\cos^2(\theta) - \sin^2(\theta)).$$

Using double angle identities yields

$$\sin(2\theta)(C - A) = B\cos(2\theta),$$

which is simply either $\tan(2\theta) = \dfrac{B}{C - A}$, or $\cot(2\theta) = \dfrac{C - A}{B}$.

For the conic $4x^2 + 6xy - 2y^2 + 7x + y - 1 = 0$, $A = 4$, $B = 6$ and $C = -2$. Solving $\tan(2\theta) = \dfrac{B}{C - A}$ for these values of $A$, $B$ and $C$ gives $\tan(2\theta) = -1$, or $\theta = \dfrac{\pi}{8}$. This gives the following values for the transformed coefficients:

$$\begin{aligned}
A' &= 1 - 3\sqrt{2}\\
B' &= 0\\
C' &= 1 + 3\sqrt{2}\\
D' &= 7\,\cos\left(\frac{3\pi}{8}\right) - \sin\left(\frac{3\pi}{8}\right)\\
E' &= 7\,\sin\left(\frac{3\pi}{8}\right) + \cos\left(\frac{3\pi}{8}\right)\\
F' &= -1,
\end{aligned}$$

which gives the new conic section to be

$$\left(1-3\sqrt{2}\right)x'^{\,2}+\left(1+3\sqrt{2}\right)y'^{\,2}+\left(7\cos\left(\frac{3\pi}{8}\right)-\sin\left(\frac{3\pi}{8}\right)\right)x'$$
$$+\left(7\sin\left(\frac{3\pi}{8}\right)+\cos\left(\frac{3\pi}{8}\right)\right)y'-1=0.$$

We plot both the original conic and the rotated conic below:

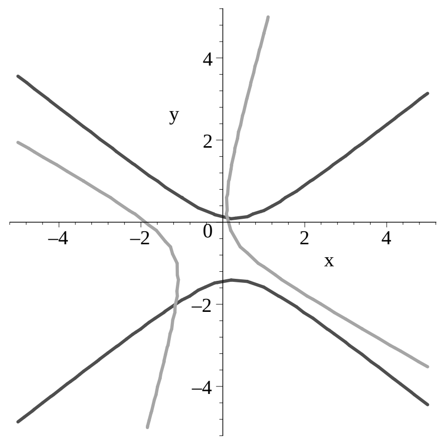


Figure 4.2: Conic section, and rotated conic section. The hyperbola opening straight up and down is the transformed $(x', y')$ hyperbola, since $B' = 0$.

## 4.2 Applications of Linear Systems to Curve Fitting

1. If a set of data has two points that have the same $x$-coordinate, but different $y$-coordinates, a problem occurs when attempting to perform Gauss-Jordan elimination. As an example, consider the function $y = Ax^2 + Bx + C$, and the set of points $\{(1,1),(2,3),(1,2)\}$. What is the problem and why does it occur?

The system will be inconsistent. Consider the two points $(x_1, y_1)$ and $(x_1, y_2)$. In order to fit these points to a function of the form

$$y = a_1 f_1(x) + a_2 f_2(x) + \cdots + a_n f_n(x),$$

we would have to plug in the points $(x_1, y_1)$ and $(x_1, y_2)$. This results in the two equations

$$\begin{aligned} y_1 &= a_1 f_1(x_1) + a_2 f_2(x_1) + \cdots + a_n f_n(x_1) \\ y_2 &= a_1 f_1(x_1) + a_2 f_2(x_1) + \cdots + a_n f_n(x_1). \end{aligned}$$

Clearly this will cause a problem when row reducing.

Let us look at the example. The system of equations that we arrive at is

$$\begin{aligned} A + B + C &= 1 \\ 4A + 2B + C &= 3 \\ A + B + C &= 2, \end{aligned}$$

which is augmented matrix form is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 2 & 1 & 3 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

Upon row reducing we get

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Notice that the last equations reads $0 = 1$, hence the system has no solution.

2. Set up, but do not solve, the matrix required to find the constants to fit the following data points to the corresponding functions.

(a) $\{(0,0), (1,2), (-3,4)\}$, $\left\{1, x, x^2\right\}$

If we set $y = Ax^2 + Bx + C$, the our system of equations is

$$\begin{aligned} C &= 0 \\ A + B + C &= 2 \\ 9A - 3B + C &= 4. \end{aligned}$$

In augmented matrix form this system is

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \\ 9 & -3 & 1 & 4 \end{bmatrix}.$$

(b) $\{(0,0),(1,2),(-3,4),(-1,5)\}$, $\{1,x,x^2,x^3\}$

If we set $y = Ax^3 + Bx^2 + Cx + D$, the our system of equations is

$$\begin{aligned} D &= 0 \\ A + B + C + D &= 2 \\ -27A + 9B - 3C + D &= 4 \\ -A + B - C + D &= 5. \end{aligned}$$

In augmented matrix form this system is

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 2 \\ -27 & 9 & -3 & 1 & 4 \\ -1 & 1 & -1 & 1 & 5 \end{bmatrix}.$$

(c) $\{(2,3),(1,2),(-3,4),(-1,5)\}$, $\left\{1,x,x^2,\frac{1}{x}\right\}$

If we set $y = \frac{A}{x} + Bx^2 + Cx + D$, the our system of equations is

$$\begin{aligned} \frac{1}{2}A + 4B + 2C + D &= 3 \\ A + B + C + D &= 2 \\ -\frac{1}{3}A + 9B - 3C + D &= 4 \\ -A + B - C + D &= 5. \end{aligned}$$

In augmented matrix form this system is

$$\begin{bmatrix} \frac{1}{2} & 4 & 2 & 1 & 3 \\ 1 & 1 & 1 & 1 & 2 \\ -\frac{1}{3} & 9 & -3 & 1 & 4 \\ -1 & 1 & -1 & 1 & 5 \end{bmatrix}.$$

(d) $\left\{(0,1),(\pi,2),\left(-\frac{\pi}{4},-1\right)\right\}$, $\{1,\sin(x),\cos(x)\}$

If we set $y = A\cos(x) + B\sin(x) + C$, then our system of equations becomes

$$\begin{aligned} A + C &= 1 \\ -A + C &= 2 \\ \frac{1}{\sqrt{2}}A - \frac{1}{\sqrt{2}}B + C &= -1. \end{aligned}$$

In augmented matrix form this system is

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 1 & -1 \end{bmatrix}.$$

3. So far, this section has been devoted to finding a one-dimensional curve of the form $y = a_1 f_1(x) + a_2 f_2(x) + \cdots + a_n f_n(x)$ given a set of $n$ data points of the form $(x_i, y_i)$. Discuss how this method can be extended to functions of two variables, given by $z = a_1 f_1(x, y) + a_2 f_2(x, y) + \cdots + a_n f_n(x, y)$, with $n$ data points of the form $(x_i, y_i, z_i)$?

We simply start with a set of equations of the form

$$z_i = a_1 f_1(x_i, y_i) + a_2 f_2(x_i, y_i) + \cdots + a_n f_n(x_i, y_i),$$

for $i \leq i \leq n$. Then, all we do is modify equation (4.7) as follows:

$$\begin{bmatrix} f_1(x_1, y_1) & f_2(x_1, y_1) & \cdots & f_n(x_1, y_1) & z_1 \\ f_1(x_2, y_2) & f_2(x_2, y_2) & \cdots & f_n(x_2, y_2) & z_2 \\ \vdots & \vdots & & \vdots & \vdots \\ f_1(x_n, y_n) & f_2(x_n, y_n) & \cdots & f_n(x_n, y_n) & z_n \end{bmatrix},$$

and row reduce to find the values of the $a_i's$.

4. Set up, but do not solve, the matrix required to find the constants to fit the following data points to the corresponding functions:

(a) $\{(0, 1, 2), (1, 2, 4), (-1, 2, -1), (1, 0, -3)\}$, $\{1, x, y, xy\}$

Now we set $z = Axy + By + Cx + D$, to get the system of equations

$$\begin{aligned} B + D &= 2 \\ 2A + 2B + C + D &= 4 \\ -2A + 2B - C + D &= -1 \\ C + D &= -3, \end{aligned}$$

which in augmented matrix form is

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 2 \\ 2 & 2 & 1 & 1 & 4 \\ -2 & 2 & -1 & 1 & -1 \\ 0 & 0 & 1 & 1 & -3 \end{bmatrix}.$$

(b) $\{(0,1,2),(1,2,4),(-1,2,-1),(1,0,-3)\}$, $\left\{1,x,y,(x-y)^2\right\}$

For our second set of points, we set our function to $z = A(x-y)^2 + By + Cx + D$, to get the system of equations

$$\begin{aligned} A + B + D &= 2 \\ A + 2B + C + D &= 4 \\ 9A + 2B - C + D &= -1 \\ A + C + D &= -3, \end{aligned}$$

which in augmented matrix form is

$$\left[\begin{array}{ccccc} 1 & 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 1 & 4 \\ 9 & 2 & -1 & 1 & -1 \\ 1 & 0 & 1 & 1 & -3 \end{array}\right].$$

5. The Lagrange polynomial $L(x)$ for a data set $D_n$ of $n$ points given by

$$\{(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)\}$$

for distinct $x$-coordinates, is the smallest degree polynomial that passes through all of the points of the data set. What is the maximum degree of the Lagrange polynomial $L(x)$ passing through $D_n$?

So we are looking for the largest of the smallest degree polynomial that will pass through all the points in $D_n$. From our observations thus far, a line is the smallest degree polynomial that normally passes through two points, a parabola through three points etc... In general, what we find is that the maximum degree of $L(x)$ is $n-1$.

## 4.3 Applications of Linear Systems to Economics

There are no Homework Problems for this section.

## 4.4 Applications of Matrix Multiplication to Geometry

1. Consider the point $P(3,0)$. Without using matrix multiplication, find the resulting points $Q$, $R$, and $S$ after rotating $P$ about the origin by angles $\frac{\pi}{4}$, $\frac{\pi}{2}$, and $\frac{3}{2}\pi$, respectively.

$$Q\left(\tfrac{3}{\sqrt{2}}, \tfrac{3}{\sqrt{2}}\right) \quad R\,(0,3) \quad S\,(0,-3)$$

2. Use matrix multiplication to perform the rotations in problem 1.

$$Q\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$R\,(0,3) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$S\,(0,-3) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

3. Given a point $P$, let $Q$ be the point corresponding to the rotation of $P$ about the origin through an angle $\theta$. Let $R$ be the point corresponding to the rotation of $Q$ about the origin through the angle $\phi$. Verify that

$$A_\phi A_\theta = A_{\phi+\theta}$$

and thus that

$$R = A_{\phi+\theta} P = A_\phi A_\theta P = A_\theta A_\phi P$$

We start with simple matrix multiplication and sum of angle trig identities for cos and sin:

$$\begin{aligned} A_\phi A_\theta &= \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\phi)\cos(\theta) - \sin(\phi)\sin(\theta) & -\cos(\phi)\sin(\theta) - \sin(\phi)\cos(\theta) \\ \sin(\phi)\cos(\theta) + \cos(\phi)\sin(\theta) & \cos(\phi)\cos(\theta) - \sin(\phi)\sin(\theta) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\phi+\theta) & -\sin(\phi+\theta) \\ \sin(\phi+\theta) & \cos(\phi+\theta) \end{bmatrix} = A_{\phi+\theta}. \end{aligned}$$

4. Geometrically, the same property discussed in problem 3 should hold for an arbitrary center of rotation. For instance, if we start with a point $P$, rotate

it through an angle $\theta$ to the point $Q$, and the rotate $Q$ through an angle $\phi$ to end up at $R$; this should be equivalent to starting at $P$ and rotating through an angle of $\phi+\theta$, independent of the center. To show this, consider

$$Q = A_\theta(P-C)+C, \quad R = A_\theta(Q-C)+C$$

and prove that

$$A_{\phi+\theta}(P-C)+C = A_\phi\left[Q-C\right]+C$$

The following proves the idea:

$$\begin{aligned} R &= A_\phi\left[Q-C\right]+C \\ &= A_\phi\left[A_\theta(P-C)+C)-C\right]+C \\ &= A_\phi\left[A_\theta(P-C)\right]+C \\ &= A_\phi A_\theta\left[(P-C)\right]+C \\ &= A_{\phi+\theta}\left[(P-C)\right]+C \end{aligned}$$

5. Find the coordinates of the point $Q$ corresponding to the point $P(3,3)$ that has been rotated about the point $C(1,1)$ by an angle of $\theta = \dfrac{\pi}{4}$.

$$\begin{aligned} Q &= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\left(\begin{bmatrix} 3 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 2\sqrt{2}+1 \end{bmatrix}. \end{aligned}$$

Therefore, our rotated point is $Q\left(1, 2\sqrt{2}+1\right)$.

6. Given a point $P$, a point $Q$ and a center of rotation $C$, how can one find the angle $\theta$ through which $P$ was rotated to end up at $Q$?

If we start with $Q = A_\theta(P-C)+C$, where the only unknown is $\theta$, then we have two equations in the unknowns $\cos(\theta)$ and $\sin(\theta)$. Solving for $\cos(\theta)$ and $\sin(\theta)$ and then taking trig inverses allows one to solve for $\theta$.

7. Consider the points $P(4,5)$ and $Q(2, 2\sqrt{2}+3)$ and center of rotation $C(2,3)$. Determine the angle $\theta$ through which $P$ was rotated about $C$ to end up at point $Q$.

We start with

$$\begin{bmatrix} 2 \\ 2\sqrt{2}+3 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\left(\begin{bmatrix} 4 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) + \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

which simplifies to

$$\begin{bmatrix} 0 \\ 2\sqrt{2} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

This yields two equations:

$$\begin{aligned} 0 &= 2\cos(\theta) - 2\sin(\theta) \\ 2\sqrt{2} &= 2\cos(\theta) + 2\sin(\theta). \end{aligned}$$

Clearly, $\theta = \frac{\pi}{4}$ is the solution to these equations.

8. Find the coordinates of the point $Q$ corresponding to the point $P(3,3)$ after it has been rotated about the point $C(1,3)$ by an angle of $\theta = \pi$. Consider this problem from a geometric point of view, explain how you could have known the answer without performing any matrix multiplication.

Since the $y$ coordinates are the same, rotation by $\theta = \pi$ is simply going to place the point on the opposite side of the center, at $Q(-1,3)$.

9. As discussed in this section, the matrix $A_\theta$ corresponds to a counter-clockwise rotation about the origin. How can you modify the matrix $A_\theta$ to perform clockwise rotations?

We would simply replace $\theta$ with $-\theta$ to get:

$$A_{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = A_\theta^{-1}.$$

10. The process of rotation about a point can be generalized to three dimensions. Given a point $P(x_0, y_0, z_0)$, determine what rotations the following matrices perform upon the point $P$.

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$A_2 = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$A_3 = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A_1$ is rotation about $x-$axis, $A_2$ about $y-$axis and $A_3$ about $z-$axis.

11. (a) What $3 \times 3$ matrix $R$ will carry out, by a single matrix multiplication by R, the following three consecutive rotations in space in the given order: first, rotate in space by the angle $\alpha$ about the $x$-axis followed by a rotation by the angle $\beta$ about the $y$-axis followed by a rotation by the angle $\gamma$ about the $z$-axis?

We simply perform the following matrix multiplication $A_3(\gamma)A_2(\beta)A_1(\alpha)$ as follows:

$$\begin{bmatrix} \cos(\gamma) & \sin(\gamma) & 0 \\ -\sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\beta) & 0 & -\sin(\beta) \\ 0 & 1 & 0 \\ \sin(\beta) & 0 & \cos(\beta) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{bmatrix} =$$

$$\begin{bmatrix} \cos(\gamma)\cos(\beta) & \sin(\gamma)\cos(\alpha) + \cos(\gamma)\sin(\beta)\sin(\alpha) & \sin(\gamma)\sin(\alpha) - \cos(\gamma)\sin(\beta)\cos(\alpha) \\ -\sin(\gamma)\cos(\beta) & \cos(\gamma)\cos(\alpha) - \sin(\gamma)\sin(\beta)\sin(\alpha) & \cos(\gamma)\sin(\alpha) + \sin(\gamma)\sin(\beta)\cos(\alpha) \\ \sin(\beta) & -\cos(\beta)\sin(\alpha) & \cos(\beta)\cos(\alpha) \end{bmatrix}$$

(b) Is it the same matrix $R$ if we switch the order of these three consecutive rotations, explain?

No, order does matter, and simple calculations prove this. If one considers the geometric interpretations, the answer should be self-evident.

12. How can you use matrix multiplication and addition/subtraction to rotate in space about a line parallel to one of the three coordinate axes?

Given a point $P$ and a line $L$ in the direction of one of the coordinate axes simply perform a shift similar to that which we have done in the past, to result in the point $Q$:

$$Q = A_k\,(P - C) + C.$$

Here, $A_k$, for $1 \le k \le 3$, is one of the three rotation matrices from the previous problem. The definition of the point $C$ is not the same, as there is no center of rotation, instead there is a rotation about the line $L$. In this instance, $C$ is the point whose only nonzero coordinate coordinate of $P$ corresponding to the axis of rotation. This process simply creates a plane of rotation which happens to be one $xy$, $xz$ or $yz$ coordinate planes.

As a quick example, if we wish to rotate about the line parallel to the $x$-axis, passing through the point $(1, 2, 3)$, then $C = (1, 0, 0)$.

13. Using the information learned in this section, do (or redo) *Homework* problem 7 of Section 4.1.

See the corresponding problem from Section 4.1 for a full explanation.

## 4.5 An Application of Matrix Multiplication to Economics

There are no Homework Problems for this section.

# Chapter 5

# Determinants, Inverses, and Cramer's Rule

## 5.1 Determinants and Inverses from the Adjoint Formula

1. Compute the transpose of the following matrices.

| | matrix | transpose |
|---|---|---|
| (a) | $\begin{bmatrix} 1 & 3 \\ -2 & -5 \end{bmatrix}$ | $\begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix}$ |
| (b) | $\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$ | $\begin{bmatrix} -2 & 0 & 3 \end{bmatrix}$ |
| (c) | $\begin{bmatrix} 2 & 0 & -1 \\ 2 & -4 & 4 \end{bmatrix}$ | $\begin{bmatrix} 2 & 2 \\ 0 & -4 \\ -1 & 4 \end{bmatrix}$ |
| (d) | $\begin{bmatrix} 2 & -4 \\ -1 & 0 \\ 8 & 5 \end{bmatrix}$ | $\begin{bmatrix} 2 & -1 & 8 \\ -4 & 0 & 5 \end{bmatrix}$ |
| (e) | $\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ | $\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ |

(f) $\begin{bmatrix} -2 & 1 & 9 \\ -4 & 2 & 8 \\ -1 & 4 & -5 \end{bmatrix}$ $\begin{bmatrix} -2 & -4 & -1 \\ 1 & 2 & 4 \\ 9 & 8 & -5 \end{bmatrix}$

(g) $\begin{bmatrix} 0 & -4 & -8 \\ 4 & 0 & -5 \\ 8 & 5 & 0 \end{bmatrix}$ $\begin{bmatrix} 2 & 4 & 8 \\ -4 & 0 & 5 \\ -8 & -5 & 3 \end{bmatrix}$

(h) $\begin{bmatrix} -1 & 0 & 1 & 2 \\ 7 & -1 & 2 & -4 \\ 1 & 4 & 1 & 2 \end{bmatrix}$ $\begin{bmatrix} -1 & 7 & 1 \\ 0 & -1 & 4 \\ 1 & 2 & 1 \\ 2 & -4 & 2 \end{bmatrix}$

(i) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

2. A matrix is *symmetric* if $A^T = A$. Which of the matrices from problem 1 are symmetric?

The matrix from part (e) is symmetric.

3. A matrix is *antisymmetric* if $A^T = -A$. Which of the matrices from problem 1 are anti-symmetric?

The matrix from part g) is anti-symmetric.

4. Compute the determinants of the following matrices:

(a) $\det\left(\begin{bmatrix} 2 & -3 \\ 8 & -4 \end{bmatrix}\right) = 16$ (b) $\det\left(\begin{bmatrix} 2 & -2 \\ 5 & 1 \end{bmatrix}\right) = 12$

(c) $\det\left(\begin{bmatrix} 2 & -8 \\ -4 & 16 \end{bmatrix}\right) = 0$ (d) $\det\left(\begin{bmatrix} \frac{3}{5} & -\frac{1}{10} \\ 8 & -\frac{4}{3} \end{bmatrix}\right) = 0$

(e) $\det\left(\begin{bmatrix} 0 & -\frac{2}{3} \\ 5 & \frac{1}{10} \end{bmatrix}\right) = \frac{10}{3}$ (f) $\det\left(\begin{bmatrix} 1 & 3 & 0 \\ -4 & 16 & 3 \\ 0 & -3 & -5 \end{bmatrix}\right) = -131$

(g) $\det\left(\begin{bmatrix} 1 & -3 & 1 \\ 9 & -4 & 0 \\ -3 & 5 & 2 \end{bmatrix}\right) = 79$ (h) $\det\left(\begin{bmatrix} 2 & -3 & 8 \\ -4 & 0 & 1 \\ 5 & -2 & 4 \end{bmatrix}\right) = 5$

(i) $\det\left(\begin{bmatrix} 1 & -3 & -5 \\ 5 & 4 & 5 \\ -1 & 3 & 5 \end{bmatrix}\right) = 0$

5. Compute the cofactor matrix to each of the matrices from problem 4.

| | matrix | cofactor matrix |
|---|---|---|
| (a) | $\begin{bmatrix} 2 & -3 \\ 8 & -4 \end{bmatrix}$ | $\begin{bmatrix} -4 & -8 \\ 3 & 2 \end{bmatrix}$ |
| (b) | $\begin{bmatrix} 2 & -2 \\ 5 & 1 \end{bmatrix}$ | $\begin{bmatrix} 1 & -5 \\ 2 & 2 \end{bmatrix}$ |
| (c) | $\begin{bmatrix} 2 & -8 \\ -4 & 16 \end{bmatrix}$ | $\begin{bmatrix} 16 & 4 \\ 8 & 2 \end{bmatrix}$ |
| (d) | $\begin{bmatrix} \frac{3}{5} & -\frac{1}{10} \\ 8 & -\frac{4}{3} \end{bmatrix}$ | $\begin{bmatrix} -\frac{4}{3} & -8 \\ \frac{1}{10} & \frac{3}{5} \end{bmatrix}$ |
| (e) | $\begin{bmatrix} 0 & -\frac{2}{3} \\ 5 & \frac{1}{10} \end{bmatrix}$ | $\begin{bmatrix} \frac{1}{10} & -5 \\ \frac{2}{3} & 0 \end{bmatrix}$ |
| (f) | $\begin{bmatrix} 1 & 3 & 0 \\ -4 & 16 & 3 \\ 0 & -3 & -5 \end{bmatrix}$ | $\begin{bmatrix} -71 & -20 & 12 \\ 15 & -5 & 3 \\ 9 & -3 & 28 \end{bmatrix}$ |
| (g) | $\begin{bmatrix} 1 & -3 & 1 \\ 9 & -4 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ | $\begin{bmatrix} -8 & -18 & 33 \\ 11 & 5 & 4 \\ 4 & 9 & 23 \end{bmatrix}$ |
| (h) | $\begin{bmatrix} 2 & -3 & 8 \\ -4 & 0 & 1 \\ 5 & -2 & 4 \end{bmatrix}$ | $\begin{bmatrix} 2 & 21 & 8 \\ -4 & -32 & -11 \\ -3 & -34 & -12 \end{bmatrix}$ |
| (i) | $\begin{bmatrix} 1 & -3 & -5 \\ 5 & 4 & 5 \\ -1 & 3 & 5 \end{bmatrix}$ | $\begin{bmatrix} 5 & -30 & 19 \\ 0 & 0 & 0 \\ 5 & -30 & 19 \end{bmatrix}$ |

6. Compute the inverse matrix to each of the matrices from problem 4, using the cofactor matrices from problem 5.

| | matrix | matrix inverse |
|---|---|---|
| (a) | $\begin{bmatrix} 2 & -3 \\ 8 & -4 \end{bmatrix}$ | $\begin{bmatrix} -\frac{1}{4} & \frac{3}{16} \\ -\frac{1}{2} & \frac{1}{8} \end{bmatrix}$ |
| (b) | $\begin{bmatrix} 2 & -2 \\ 5 & 1 \end{bmatrix}$ | $\begin{bmatrix} \frac{1}{12} & \frac{1}{6} \\ -\frac{5}{12} & \frac{1}{6} \end{bmatrix}$ |
| (c) | $\begin{bmatrix} 2 & -8 \\ -4 & 16 \end{bmatrix}$ | not invertible |
| d) | $\begin{bmatrix} \dfrac{3}{5} & -\dfrac{1}{10} \\ 8 & -\dfrac{4}{3} \end{bmatrix}$ | not invertible |
| (e) | $\begin{bmatrix} 0 & -\dfrac{2}{3} \\ 5 & \dfrac{1}{10} \end{bmatrix}$ | $\begin{bmatrix} \frac{3}{100} & \frac{1}{5} \\ -\frac{3}{2} & 0 \end{bmatrix}$ |
| (f) | $\begin{bmatrix} 1 & 3 & 0 \\ -4 & 16 & 3 \\ 0 & -3 & -5 \end{bmatrix}$ | $\begin{bmatrix} \frac{71}{131} & -\frac{15}{131} & -\frac{9}{131} \\ \frac{20}{131} & \frac{5}{131} & \frac{3}{131} \\ -\frac{12}{131} & -\frac{3}{131} & -\frac{28}{131} \end{bmatrix}$ |
| (g) | $\begin{bmatrix} 1 & -3 & 1 \\ 9 & -4 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ | $\begin{bmatrix} -\frac{8}{79} & \frac{11}{79} & \frac{4}{79} \\ -\frac{18}{79} & \frac{5}{79} & \frac{9}{79} \\ \frac{33}{79} & \frac{4}{79} & \frac{23}{79} \end{bmatrix}$ |
| (h) | $\begin{bmatrix} 2 & -3 & 8 \\ -4 & 0 & 1 \\ 5 & -2 & 4 \end{bmatrix}$ | $\begin{bmatrix} \frac{2}{5} & -\frac{4}{5} & -\frac{3}{5} \\ \frac{21}{5} & -\frac{32}{5} & -\frac{34}{5} \\ \frac{8}{5} & -\frac{11}{5} & -\frac{12}{5} \end{bmatrix}$ |
| (i) | $\begin{bmatrix} 1 & -3 & -5 \\ 5 & 4 & 5 \\ -1 & 3 & 5 \end{bmatrix}$ | not invertible |

7. Use your answers (if possible) to problem 6 to help solve the following systems:

(a)
$$\begin{array}{c} 2x - 3y = 6 \\ 8x - 4y = 4 \end{array} \longrightarrow \begin{bmatrix} 2 & -3 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{16} \\ -\frac{1}{2} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} \\ -\frac{5}{2} \end{bmatrix}$$

(b)
$$\begin{array}{c} 2x - 2y = 7 \\ 5x + y = 8 \end{array} \longrightarrow \begin{bmatrix} 2 & -2 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{12} & \frac{1}{6} \\ -\frac{5}{12} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{23}{12} \\ -\frac{19}{12} \end{bmatrix}$$

(c)
$$\begin{array}{c} 2x - 3y = -1 \\ 8x - 4y = 3 \end{array} \longrightarrow \begin{bmatrix} 2 & -3 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{16} \\ -\frac{1}{2} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{13}{16} \\ \frac{7}{8} \end{bmatrix}$$

(d)
$$\begin{array}{c} 2x - 2y = 6 \\ 5x + y = -5 \end{array} \longrightarrow \begin{bmatrix} 2 & -2 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{12} & \frac{1}{6} \\ -\frac{5}{12} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ -\frac{10}{3} \end{bmatrix}$$

(e)
$$\begin{array}{c} x - 3y + z = 1 \\ 9x - 4y = 4 \\ -3x + 5y + 2z = 1 \end{array} \longrightarrow \begin{bmatrix} 1 & -3 & 1 \\ 9 & -4 & 0 \\ -3 & 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{8}{79} & \frac{11}{79} & \frac{4}{79} \\ -\frac{18}{79} & \frac{5}{79} & \frac{9}{79} \\ \frac{33}{79} & \frac{4}{79} & \frac{23}{79} \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{40}{79} \\ \frac{11}{79} \\ \frac{72}{79} \end{bmatrix}$$

(f)
$$\begin{array}{c} 2x - 3y + 8z = 3 \\ -4x + z = 5 \\ 5x - 2y + 4z = 6 \end{array} \longrightarrow \begin{bmatrix} 2 & -3 & 8 \\ -4 & 0 & 1 \\ 5 & -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & -\frac{4}{5} & -\frac{3}{5} \\ \frac{21}{5} & -\frac{32}{5} & -\frac{34}{5} \\ \frac{8}{5} & -\frac{11}{5} & -\frac{12}{5} \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{32}{5} \\ -\frac{301}{5} \\ -\frac{103}{5} \end{bmatrix}$$

(g)
$$\begin{aligned} x - 3y + z &= 8 \\ 9x - 4y &= -2 \\ -3x + 5y + 2z &= 3 \end{aligned} \quad \to \quad \begin{bmatrix} 1 & -3 & 1 \\ 9 & -4 & 0 \\ -3 & 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{8}{79} & \frac{11}{79} & \frac{4}{79} \\ -\frac{18}{79} & \frac{5}{79} & \frac{9}{79} \\ \frac{33}{79} & \frac{4}{79} & \frac{23}{79} \end{bmatrix} \begin{bmatrix} 8 \\ -2 \\ 3 \end{bmatrix} \quad \to \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 26 \\ 10 \end{bmatrix}$$

(h)
$$\begin{aligned} x - 3y - 5z &= 2 \\ 5x + 4y + 5z &= -1 \\ -x + 3y + 5z &= 0 \end{aligned} \quad \to \quad \text{no solution!}$$

8. Determine values of $\lambda$ such that the following matrices are not invertible. The values of $\lambda$ that make each of the following matrices singular are called *eigenvalues*. In general, eigenvalues are found by solving for $\lambda$ the equation $\det(A - \lambda I_n) = 0$, for $A \in \mathbb{R}^{n \times n}$.

(a) $\begin{bmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix}$ (b) $\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$ (c) $\begin{bmatrix} -\lambda & 3 & 4 \\ 4 & -4-\lambda & -8 \\ 6 & -9 & -10-\lambda \end{bmatrix}$

The only root of the above polynomial is $\lambda = 2$.

$$\text{b) } \det\left(\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}\right) = (3-\lambda)(3-\lambda) - 1 = \lambda^2 - 6\lambda + 8.$$

The two roots of the above polynomial are $\lambda = 2$ and $\lambda = 4$.

$$\begin{aligned} \text{c) } \det\left(\begin{bmatrix} -\lambda & 3 & 4 \\ 4 & -4-\lambda & -8 \\ 6 & -9 & -10-\lambda \end{bmatrix}\right) &= -\lambda\left((-4-\lambda)(-10-\lambda) - 72\right) \\ &- 3\left(4(-10-\lambda) + 48\right) + 4\left(-36 - 6(-4-\lambda)\right) \\ &= -(\lambda+18)(\lambda-2)^2 \end{aligned}$$

The two roots of the above polynomial are $\lambda = 2$ and $\lambda = -18$.

9. A matrix $A$ is *diagonal* if $A_{i,j} = 0$ for $i \neq j$. Entries on the diagonal are not required to be nonzero, however, for this problem, assume that $A_{i,i} \neq 0$ for $1 \leq i \leq n$. Show that the inverse matrix to $A$ is a diagonal matrix with entries $\frac{1}{A_{i,i}}$.

Let $B$ be the matrix with diagonal entries $\frac{1}{A_{i,i}}$ and zeros elsewhere. We simply perform matrix multiplication $AB$ and show that $AB = I_n$, for $I_n$ the $n \times n$ identity matrix:

$$\begin{bmatrix} A_{1,1} & 0 & \cdots & 0 \\ 0 & A_{2,2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & A_{n,n} \end{bmatrix} \begin{bmatrix} \frac{1}{A_{1,1}} & 0 & \cdots & 0 \\ 0 & \frac{1}{A_{2,2}} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{A_{n,n}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

One can similarly show that $BA = I_n$. This by definition implies that $B = A^{-1}$.

We could have also argued that the product of two diagonal matrices is another diagonal matrix whose entries along the diagonal are simply the products of the corresponding entries on the diagonals of $A$ and $B$, which gives $A_{i,i}\frac{1}{A_{i,i}} = 1$ for $1 \leq i \leq n$.

10. A matrix $A$ is *upper triangular* if $A_{i,j} = 0$ for $i > j$, and is *lower triangular* of $A_{i,j} = 0$ for $i < j$. Is the inverse of a lower/upper triangular matrix $D$ also a lower/upper triangular matrix?

We know that the product of two upper triangular matrices is upper triangular, the same goes with the product of two lower triangular matrices. The identity matrix, by definition, is both upper triangular and lower triangular.

If $E$ is the inverse of a triangular matrix $D$, then by definition, $ED = DE = I_n$. If $D$ is upper triangular, then $D_{i,j} = 0$ for $i > j$, and

$$\begin{aligned}(ED)_{i,j} &= \sum_{k=1}^{n} E_{i,k} D_{k,j} \\ &= \sum_{k=1}^{j} E_{i,k} D_{k,j}\end{aligned}$$

Similarly,

$$\begin{aligned}(DE)_{i,j} &= \sum_{k=1}^{n} D_{i,k} E_{k,j} \\ &= \sum_{k=j}^{n} D_{i,k} E_{k,j}.\end{aligned}$$

For $(DE)_{i,j} = (ED)_{i,j} = 0$ for $i \neq j$, for an arbitrary upper triangular matrix $D$, then to satisfy the above two conditions, we see that $E_{i,j} = 0$ for $i > j$.

A similar argument can be made for lower triangular matrices.

11. Compute the inverses of the following matrices:

| | matrix | matrix inverse |
| --- | --- | --- |
| (a) | $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ | $\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}$ |
| (b) | $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ | $\begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{9} \end{bmatrix}$ |
| (c) | $\begin{bmatrix} \frac{5}{7} & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{2}{3} \end{bmatrix}$ | $\begin{bmatrix} \frac{7}{5} & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & \frac{3}{2} \end{bmatrix}$ |

12. A matrix $A$ is *orthogonal* if its transpose is equal to its inverse, that is, $A^{-1} = A^T$. Explain why a symmetric or anti-symmetric or orthogonal matrix must be square.

In order for there to be a direct relationship between a matrix and its transpose or inverse, the matrix must be square. I.e. if $A^T = A$, then their dimensions must agree. Furthermore, inverse matrices exist for only square matrices, hence orthogonal matrices must be square.

13. Let $A$ be a square matrix. Show that $A + A^T$ is symmetric while $A - A^T$ is antisymmetric.

First we show that $A + A^T$ is symmetric, i.e. $\left(A + A^T\right)^T = A + A^T$:

$$\begin{aligned} \left(A + A^T\right)^T &= A^T + \left(A^T\right)^T \\ &= A^T + A \\ &= A + A^T. \end{aligned}$$

Next we show that $A - A^T$ is anti-symmetric, i.e. $\left(A - A^T\right)^T = -\left(A - A^T\right)$:

$$\begin{aligned} \left(A - A^T\right)^T &= A^T - \left(A^T\right)^T \\ &= A^T - A \\ &= -A + A^T \\ &= -\left(A - A^T\right). \end{aligned}$$

14. Let $A$ be a square matrix. Show that $A$ can be written as the sum of a symmetric and an antisymmetric matrix.

Using problem 13, notice that

$$A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right),$$

so setting $B = \frac{1}{2}\left(A + A^T\right)$, and $C = \frac{1}{2}\left(A - A^T\right)$, we have that $A = B + C$ where $B$ is symmetric, and $C$ is anti-symmetric.

15. Let $A$ be any matrix. Show that both $AA^T$ and $A^T A$ are symmetric matrices.

First we show that $\left(AA^T\right)^T = AA^T$:

$$\begin{aligned}\left(AA^T\right)^T &= \left(A^T\right)^T A^T \\ &= AA^T\end{aligned}$$

Secondly, we show that $\left(A^T A\right)^T = A^T A$:

$$\begin{aligned}\left(A^T A\right)^T &= A^T \left(A^T\right)^T \\ &= A^T A\end{aligned}$$

16. Explain why $(AB)^T = B^T A^T$.

This is simply due to the interaction of transpose and matrix multiplication. For instance, when we multiply $AB$, we take rows of $A$ multiplied by columns of $B$. So when we transpose, we need to multiply rows of $B$ times columns of $A$.

17. Let $n$ be any positive integer and $A$ be any invertible square matrix. Show that $(A^n)^{-1} = \left(A^{-1}\right)^n$.

If $A$ is invertible, then $AA^{-1} = I_n$. Notice that $AA = A^2$, and $A^{-1}A^{-1} = \left(A^{-1}\right)^2$ thus

$$\begin{aligned}A^2\left(A^{-1}\right)^2 &= AAA^{-1}A^{-1} \\ &= AI_nA^{-1} \\ &= AA^{-1} \\ &= I_n,\end{aligned}$$

thus $\left(A^{-1}\right)^2 = \left(A^2\right)^{-1}$. Through an inductive argument, using the above steps (for $n = 2$), it can easily be shown that$(A^n)^{-1} = \left(A^{-1}\right)^n$ for any positive arbitrary integer $n$.

18. Let $E$ be an elementary matrix. Does $E$ always have an inverse, and if so, is $E^{-1}$ also an elementary matrix?

As we have discussed before, each type of elementary matrix is invertible, and of the same type.

## 5.2 Determinants by Expanding Along Any Row or Column

1. Compute the determinants of the following matrices by expanding along the first row.

(a)
$$\begin{aligned}\det\left(\begin{bmatrix} 1 & -1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right) &= 1\det\left(\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\right) - (-1)\det\left(\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}\right) \\ &\quad + 1\det\left(\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}\right) \\ &= 1\ (0) - (-1)\ (0) + 1\ (1) \\ &= 1\end{aligned}$$

(b)
$$\begin{aligned}\det\left(\begin{bmatrix} 2 & -2 & 2 \\ -2 & -2 & 0 \\ 2 & 0 & 0 \end{bmatrix}\right) &= 2\det\left(\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}\right) - (-2)\det\left(\begin{bmatrix} -2 & 0 \\ 2 & 0 \end{bmatrix}\right) \\ &\quad + 2\det\left(\begin{bmatrix} -2 & -2 \\ 2 & 0 \end{bmatrix}\right) \\ &= 2\ (0) - (-2)\ (0) + 2\ (4) \\ &= 8\end{aligned}$$

(c)
$$\det\left(\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & -1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}\right) = 1\det\left(\begin{bmatrix} 1 & -1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\right)$$

$$
\begin{aligned}
&- (-1)\det\left(\begin{bmatrix} 0 & -1 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}\right) + 1\det\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}\right) \\
&+ 0\det\left(\begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & 1 & 0 \end{bmatrix}\right) \\
= {}& 1\left(1\det\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right) - (-1)\det\left(\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}\right)\right. \\
&\left. +1\det\left(\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}\right)\right) - (-1)\left(0\det\left(\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}\right)\right. \\
&\left. -(-1)\det\left(\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\right) + 1\det\left(\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\right)\right) \\
&+ 1\left(0\det\left(\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}\right) - 1\det\left(\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\right)\right. \\
&\left. +1\det\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)\right) \\
= {}& 1\ (1(-1) - (-1)(-1) + 1(1)) + 1\ (0 - (-1)(1) + 1(-1)) \\
&+ 1\ (0 - 1(1) + 1(0)) \\
= {}& -2
\end{aligned}
$$

(d)
$$
\begin{aligned}
\det\left(\begin{bmatrix} 3 & -2 & 2 & 1 \\ 1 & -1 & 6 & 2 \\ 2 & -1 & 0 & 0 \\ -2 & 1 & 4 & 1 \end{bmatrix}\right) &= 3\det\left(\begin{bmatrix} -1 & 6 & 2 \\ -1 & 0 & 0 \\ 1 & 4 & 1 \end{bmatrix}\right) \\
&- (-2)\det\left(\begin{bmatrix} 1 & 6 & 2 \\ 2 & 0 & 0 \\ -2 & 4 & 1 \end{bmatrix}\right) + 2\det\left(\begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 0 \\ -2 & 1 & 1 \end{bmatrix}\right) \\
&- 1\det\left(\begin{bmatrix} 1 & -1 & 6 \\ 2 & -1 & 0 \\ -2 & 1 & 4 \end{bmatrix}\right) \\
&= 3\left(-1\det\left(\begin{bmatrix} 0 & 0 \\ 4 & 1 \end{bmatrix}\right) - 6\det\left(\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}\right)\right. \\
&\left. +2\det\left(\begin{bmatrix} -1 & 0 \\ 1 & 4 \end{bmatrix}\right)\right) - (-2)\left(1\det\left(\begin{bmatrix} 0 & 0 \\ 4 & 1 \end{bmatrix}\right)\right. \\
&\left. -6\det\left(\begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix}\right) + 2\det\left(\begin{bmatrix} 2 & 0 \\ -2 & 4 \end{bmatrix}\right)\right) \\
&+ 2\left(1\det\left(\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}\right) - (-1)\det\left(\begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix}\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+2\det\left(\begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}\right)\Bigg) - 1\left(1\det\left(\begin{bmatrix} -1 & 0 \\ 1 & 4 \end{bmatrix}\right)\right. \\
&\left. -(-1)\det\left(\begin{bmatrix} 2 & 0 \\ -2 & 4 \end{bmatrix}\right) + 6\det\left(\begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}\right)\right) \\
&= 3\ (-1(0) - 6(-1) + 2(-4)) + 2\ (1(0) - 6(2) + 2(8)) \\
&\quad + 2\ (1(-1) + 1(2) + 2(0)) - 1\ (1(-4) + 1(8) + 6(0)) \\
&= 0
\end{aligned}
$$

(e) $$
\begin{aligned}
\det\left(\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}\right) &= 1\det\left(\begin{bmatrix} 6 & 7 & 8 \\ 10 & 11 & 12 \\ 14 & 15 & 16 \end{bmatrix}\right) \\
&- 2\det\left(\begin{bmatrix} 5 & 7 & 8 \\ 9 & 11 & 12 \\ 13 & 15 & 16 \end{bmatrix}\right) + 3\det\left(\begin{bmatrix} 5 & 6 & 8 \\ 9 & 10 & 12 \\ 13 & 14 & 16 \end{bmatrix}\right) \\
&- 4\det\left(\begin{bmatrix} 5 & 6 & 7 \\ 9 & 10 & 11 \\ 13 & 14 & 15 \end{bmatrix}\right) \\
&= 1\left(6\det\left(\begin{bmatrix} 11 & 12 \\ 15 & 16 \end{bmatrix}\right) - 7\det\left(\begin{bmatrix} 10 & 12 \\ 14 & 16 \end{bmatrix}\right)\right. \\
&\quad \left. +8\det\left(\begin{bmatrix} 10 & 11 \\ 14 & 15 \end{bmatrix}\right)\right) - 2\left(5\det\left(\begin{bmatrix} 11 & 12 \\ 15 & 16 \end{bmatrix}\right)\right. \\
&\quad \left. -7\det\left(\begin{bmatrix} 9 & 12 \\ 13 & 16 \end{bmatrix}\right) + 8\det\left(\begin{bmatrix} 9 & 11 \\ 13 & 15 \end{bmatrix}\right)\right) \\
&\quad + 3\left(5\det\left(\begin{bmatrix} 10 & 12 \\ 15 & 16 \end{bmatrix}\right) - 6\det\left(\begin{bmatrix} 9 & 12 \\ 13 & 16 \end{bmatrix}\right)\right. \\
&\quad \left. +8\det\left(\begin{bmatrix} 9 & 10 \\ 13 & 14 \end{bmatrix}\right)\right) - 4\left(5\det\left(\begin{bmatrix} 10 & 11 \\ 14 & 15 \end{bmatrix}\right)\right. \\
&\quad \left. -6\det\left(\begin{bmatrix} 9 & 11 \\ 13 & 15 \end{bmatrix}\right) + 7\det\left(\begin{bmatrix} 9 & 10 \\ 13 & 14 \end{bmatrix}\right)\right) \\
&= 1\ (6(-4) - 7(-8) + 8(-4)) - 2\ (5(-4) - 7(-12) + 8(-8)) \\
&\quad + 3\ (5(-20) - 6(-12) + 8(-4)) - 4\ (5(-4) - 6(-8) + 7(-4)) \\
&= 0
\end{aligned}
$$

(f) $$
\det\left(\begin{bmatrix} 3 & 6 & -1 & 3 \\ 0 & -1 & 6 & 7 \\ 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 3\det\left(\begin{bmatrix} -1 & 6 & 7 \\ 0 & 4 & 8 \\ 0 & 0 & 1 \end{bmatrix}\right)
$$

$$
\begin{aligned}
&- 6\det\left(\begin{bmatrix} 0 & 6 & 7 \\ 0 & 4 & 8 \\ 0 & 0 & 1 \end{bmatrix}\right) - 1\det\left(\begin{bmatrix} 0 & -1 & 7 \\ 0 & 0 & 8 \\ 0 & 0 & 1 \end{bmatrix}\right) \\
&- 3\det\left(\begin{bmatrix} 0 & -1 & 6 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}\right) \\
&= 3\left(-1\det\left(\begin{bmatrix} 4 & 8 \\ 0 & 1 \end{bmatrix}\right) - 6\det\left(\begin{bmatrix} 0 & 8 \\ 0 & 1 \end{bmatrix}\right)\right. \\
&\left.+7\det\left(\begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}\right)\right) - 6\left(0\det\left(\begin{bmatrix} 4 & 8 \\ 0 & 1 \end{bmatrix}\right)\right. \\
&\left.-6\det\left(\begin{bmatrix} 0 & 8 \\ 0 & 1 \end{bmatrix}\right) + 7\det\left(\begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}\right)\right) \\
&- 1\left(0\det\left(\begin{bmatrix} 0 & 8 \\ 0 & 1 \end{bmatrix}\right) + 1\det\left(\begin{bmatrix} 0 & 8 \\ 0 & 1 \end{bmatrix}\right)\right. \\
&\left.+7\det\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)\right) - 3\left(0\det\left(\begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}\right)\right. \\
&\left.+1\det\left(\begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}\right) + 6\det\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)\right) \\
&= 3\,(-1(4) - 6(0) + 7(0)) - 6\,(0(4) - 6(0) + 7(0)) \\
&\quad - 1\,(0(0) + 1(0) + 7(0)) - 3\,(0(0) + 1(0) + 6(0)) \\
&= -12
\end{aligned}
$$

2. Compute the determinants of the matrices from problem 1 by expanding along the second column.

In the following solutions, we expand each matrix, and subsequent matrix about the second column. It is not necessary to use the second column for each smaller matrix.

(a)
$$
\begin{aligned}
\det\left(\begin{bmatrix} 1 & -1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right) &= -(-1)\det\left(\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}\right) + (-1)\det\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right) \\
&\quad - 0\det\left(\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}\right) \\
&= 1\,(0) + (-1)\,(-1) - 0\,(1) \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\text{(b) } \det\left(\begin{bmatrix} 2 & -2 & 2 \\ -2 & -2 & 0 \\ 2 & 0 & 0 \end{bmatrix}\right) &= -(-2)\det\left(\begin{bmatrix} -2 & 0 \\ 2 & 0 \end{bmatrix}\right) + (-2)\det\left(\begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}\right) \\
&\quad - 0\det\left(\begin{bmatrix} 2 & 2 \\ -2 & 0 \end{bmatrix}\right) \\
&= -(-2)\ (0) + (-2)\ (-4) - 0\ (4) \\
&= 8
\end{aligned}
$$

$$
\begin{aligned}
&\text{(c) } \det\left(\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & -1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}\right) = -(-1)\det\left(\begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\right) \\
&\quad + 1\det\left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\right) - (-1)\det\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}\right) \\
&\quad + 1\det\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}\right) \\
&= -(-1)\left(-(-1)\det\left(\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\right) + (-1)\det\left(\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\right)\right. \\
&\quad \left. -0\det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)\right) + 1\left(-1\det\left(\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\right)\right. \\
&\quad \left. +(-1)\det\left(\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\right) - 0\ \det\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right)\right) \\
&\quad - (-1)\left(-1\det\left(\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\right) + (-1)\det\left(\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\right)\right. \\
&\quad \left. -0\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\right) + 1\left(-1\det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)\right. \\
&\quad \left. +(-1)\det\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right) - (-1)\ \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\right) \\
&= 1\ (1(1) - 1(1) - 0(-1)) + 1\ ((-1)(1) - 1(1) - 0(0)) \\
&\quad + 1\ ((-1)(1) - 1(1) - 0(1)) + 1\ ((-1)(-1) - 1(0) + 1(1)) \\
&= -2
\end{aligned}
$$

(d) $\det\left(\begin{bmatrix} 3 & -2 & 2 & 1 \\ 1 & -1 & 6 & 2 \\ 2 & -1 & 0 & 0 \\ -2 & 1 & 4 & 1 \end{bmatrix}\right) = -(-2)\det\left(\begin{bmatrix} 1 & 6 & 2 \\ 2 & 0 & 0 \\ -2 & 4 & 1 \end{bmatrix}\right)$

$$+ (-1)\det\left(\begin{bmatrix} 3 & 2 & 1 \\ 2 & 0 & 0 \\ -2 & 4 & 1 \end{bmatrix}\right) - (-1)\det\left(\begin{bmatrix} 3 & 2 & 1 \\ 1 & 6 & 2 \\ -2 & 4 & 1 \end{bmatrix}\right)$$

$$+ 1\det\left(\begin{bmatrix} 3 & 2 & 1 \\ 1 & 6 & 2 \\ 2 & 0 & 0 \end{bmatrix}\right)$$

$$= -(-2)\left(-6\det\left(\begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix}\right) + 0\det\left(\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\right)\right.$$

$$\left.-4\det\left(\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}\right)\right) + (-1)\left(-2\det\left(\begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix}\right)\right.$$

$$\left.+0\det\left(\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\right) - 4\det\left(\begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}\right)\right)$$

$$- (-1)\left(-2\det\left(\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\right) + 6\det\left(\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\right)\right.$$

$$\left.-4\det\left(\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\right)\right) + 1\left(-2\det\left(\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}\right)\right.$$

$$\left.+6\det\left(\begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}\right) - 0\det\left(\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\right)\right)$$

$$= 2\,(-6(2) + 0(5) - 4(-4)) - 1\,(-2(2) + 0(5) - 4(-2))$$

$$+ 1\,(-2(5) + 6(5) - 4(5)) + 1\,(-2(-4) + 6(-2) - 0(5))$$

$$= 0$$

(e) $\det\left(\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}\right) = -2\det\left(\begin{bmatrix} 5 & 7 & 8 \\ 9 & 11 & 12 \\ 13 & 15 & 16 \end{bmatrix}\right)$

$$+ 6\det\left(\begin{bmatrix} 1 & 3 & 4 \\ 9 & 11 & 12 \\ 13 & 15 & 16 \end{bmatrix}\right) - 10\det\left(\begin{bmatrix} 1 & 3 & 4 \\ 5 & 7 & 8 \\ 13 & 15 & 16 \end{bmatrix}\right)$$

$$+ 14\det\left(\begin{bmatrix} 1 & 3 & 4 \\ 5 & 7 & 8 \\ 9 & 11 & 12 \end{bmatrix}\right)$$

$$= -2\left(-7\det\left(\begin{bmatrix} 9 & 12 \\ 13 & 16 \end{bmatrix}\right) + 11\det\left(\begin{bmatrix} 5 & 8 \\ 13 & 16 \end{bmatrix}\right)\right.$$
$$\left.-15\det\left(\begin{bmatrix} 5 & 8 \\ 9 & 12 \end{bmatrix}\right)\right) + 6\left(-3\det\left(\begin{bmatrix} 9 & 12 \\ 13 & 16 \end{bmatrix}\right)\right.$$
$$\left.+11\det\left(\begin{bmatrix} 1 & 4 \\ 13 & 16 \end{bmatrix}\right) - 15\det\left(\begin{bmatrix} 1 & 4 \\ 9 & 12 \end{bmatrix}\right)\right)$$
$$-10\left(-3\det\left(\begin{bmatrix} 5 & 8 \\ 13 & 16 \end{bmatrix}\right) + 7\det\left(\begin{bmatrix} 1 & 4 \\ 13 & 16 \end{bmatrix}\right)\right.$$
$$\left.-15\det\left(\begin{bmatrix} 1 & 4 \\ 5 & 8 \end{bmatrix}\right)\right) + 14\left(-3\det\left(\begin{bmatrix} 5 & 8 \\ 9 & 12 \end{bmatrix}\right)\right.$$
$$\left.+7\det\left(\begin{bmatrix} 1 & 4 \\ 9 & 12 \end{bmatrix}\right) - 11\det\left(\begin{bmatrix} 1 & 4 \\ 5 & 8 \end{bmatrix}\right)\right)$$
$$= -2\left(-7(-12) + 11(-24) - 15(-12)\right) + 6\left(-3(-12) + 11(-12) - 15(-24)\right)$$
$$-10\left(-3(-24) + 7(-36) - 15(-12)\right) + 14\left(-3(-12) + 7(-24) - 11(-12)\right)$$
$$= 0$$

(f) $$\det\left(\begin{bmatrix} 3 & 6 & -1 & 3 \\ 0 & -1 & 6 & 7 \\ 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = -6\det\left(\begin{bmatrix} 0 & 6 & 7 \\ 0 & 4 & 8 \\ 0 & 0 & 1 \end{bmatrix}\right)$$
$$+(-1)\det\left(\begin{bmatrix} 3 & -1 & 3 \\ 0 & 4 & 8 \\ 0 & 0 & 1 \end{bmatrix}\right) - 0\det\left(\begin{bmatrix} 3 & -1 & 3 \\ 0 & 6 & 7 \\ 0 & 0 & 1 \end{bmatrix}\right)$$
$$+0\det\left(\begin{bmatrix} 3 & -1 & 3 \\ 0 & 6 & 7 \\ 0 & 4 & 8 \end{bmatrix}\right)$$
$$= -6\left(-6\det\left(\begin{bmatrix} 0 & 8 \\ 0 & 1 \end{bmatrix}\right) + 4\det\left(\begin{bmatrix} 0 & 7 \\ 0 & 1 \end{bmatrix}\right)\right.$$
$$\left.-0\det\left(\begin{bmatrix} 0 & 7 \\ 0 & 8 \end{bmatrix}\right)\right) + (-1)\left(-(-1)\det\left(\begin{bmatrix} 0 & 8 \\ 0 & 1 \end{bmatrix}\right)\right.$$
$$\left.+4\det\left(\begin{bmatrix} 3 & 3 \\ 0 & 1 \end{bmatrix}\right) - 0\det\left(\begin{bmatrix} 3 & 3 \\ 0 & 8 \end{bmatrix}\right)\right)$$
$$-0\left(-(-1)\det\left(\begin{bmatrix} 0 & 7 \\ 0 & 1 \end{bmatrix}\right) + 6\det\left(\begin{bmatrix} 3 & 3 \\ 0 & 1 \end{bmatrix}\right)\right.$$
$$\left.-0\det\left(\begin{bmatrix} 3 & 3 \\ 0 & 7 \end{bmatrix}\right)\right) + 0\left(-(-1)\det\left(\begin{bmatrix} 0 & 8 \\ 0 & 1 \end{bmatrix}\right)\right.$$

$$
\begin{aligned}
&+6\det\left(\begin{bmatrix} 3 & 3 \\ 0 & 8 \end{bmatrix}\right) - 0\det\left(\begin{bmatrix} 3 & 3 \\ 0 & 7 \end{bmatrix}\right)\Bigg) \\
&= -6\ (-6(0) + 4(0) - 0(0)) - 1\ (1(0) + 4(3) - 0(24)) \\
&\quad - 0\ (1(0) + 6(3) - 0(21)) + 0\ (1(0) + 6(24) - 0(21)) \\
&= -12
\end{aligned}
$$

3. For each of the matrices in problem 1, which row or column would be the best choice to expand upon in computing the determinant?

| | |
|---|---|
| (a) row 3 or column 2 | b) row 3 or column 2 |
| (c) column 4 | d) row 3 |
| (e) it does not matter | f) column 1 |

4. Compute the determinants of the matrices from problem 1 using the row or column that you found in problem 3.

We will only do the matrices from parts (c), (d), and (f), as the matrices from (a), (b), and (e) have already been done along the corresponding row/column from the previous two problems.

$$
\begin{aligned}
\text{(c) } \det\left(\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & -1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}\right) &= -0\det\left(\begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & 1 & 0 \end{bmatrix}\right) \\
&+ 1\det\left(\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 0 \end{bmatrix}\right) - 0\det\left(\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}\right) \\
&+ 1\det\left(\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}\right) \\
&= 1\left(1\det\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right) - (-1)\det\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)\right. \\
&\left.+0\det\left(\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\right)\right) + 1\left(1\det\left(\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}\right)\right. \\
&\left.-0\det\left(\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\right) + 1\det\left(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\right)\right) \\
&= 1\ (1(0) + 1(0) + 0(0)) + 1\ (1(-2) - 0(2) + 1(0)) \\
&= -2
\end{aligned}
$$

$$\text{(d) } \det\left(\begin{bmatrix} 3 & -2 & 2 & 1 \\ 1 & -1 & 6 & 2 \\ 2 & -1 & 0 & 0 \\ -2 & 1 & 4 & 1 \end{bmatrix}\right) = 2\det\left(\begin{bmatrix} -2 & 2 & 1 \\ -1 & 6 & 2 \\ 1 & 4 & 1 \end{bmatrix}\right)$$
$$-(-1)\det\left(\begin{bmatrix} 3 & 2 & 1 \\ 1 & 6 & 2 \\ -2 & 4 & 1 \end{bmatrix}\right) + 0\det\left(\begin{bmatrix} 3 & -2 & 1 \\ 1 & 6 & 2 \\ -2 & 1 & 1 \end{bmatrix}\right)$$
$$-0\det\left(\begin{bmatrix} 3 & -2 & 2 \\ 1 & -1 & 6 \\ -2 & 1 & 4 \end{bmatrix}\right)$$
$$= 2\left(-2\det\left(\begin{bmatrix} 6 & 2 \\ 4 & 1 \end{bmatrix}\right) - 2\det\left(\begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}\right)\right.$$
$$\left.+1\det\left(\begin{bmatrix} -1 & 6 \\ 1 & 4 \end{bmatrix}\right)\right) + 1\left(3\det\left(\begin{bmatrix} 6 & 2 \\ 4 & 1 \end{bmatrix}\right)\right.$$
$$\left.-2\det\left(\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\right) + 1\det\left(\begin{bmatrix} 1 & 6 \\ -2 & 4 \end{bmatrix}\right)\right)$$
$$= 2\left(-2(-2) - 2(-3) + 1(-10)\right) + 1\left(3(-2) - 2(5) + 1(16)\right)$$
$$= 0$$

$$\text{(f) } \det\left(\begin{bmatrix} 3 & 6 & -1 & 3 \\ 0 & -1 & 6 & 7 \\ 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 3\det\left(\begin{bmatrix} -1 & 6 & 7 \\ 0 & 4 & 8 \\ 0 & 0 & 1 \end{bmatrix}\right)$$
$$-0\det\left(\begin{bmatrix} 6 & -1 & 3 \\ 0 & 4 & 8 \\ 0 & 0 & 1 \end{bmatrix}\right) + 0\det\left(\begin{bmatrix} 6 & -1 & 3 \\ -1 & 6 & 7 \\ 0 & 0 & 1 \end{bmatrix}\right)$$
$$-0\det\left(\begin{bmatrix} 6 & -1 & 3 \\ -1 & 6 & 7 \\ 0 & 4 & 8 \end{bmatrix}\right)$$
$$= 3\left(-1\det\left(\begin{bmatrix} 4 & 8 \\ 0 & 1 \end{bmatrix}\right) - 0\det\left(\begin{bmatrix} 6 & 7 \\ 0 & 1 \end{bmatrix}\right)\right.$$
$$\left.+0\det\left(\begin{bmatrix} 6 & 7 \\ 4 & 8 \end{bmatrix}\right)\right)$$
$$= 3\left(-1(4) - 0(6) + 0(20)\right)$$
$$= -12$$

5. Which of the matrices from problem 1 are singular?

The matrices from parts (d) and (e) have determinant zero, and are therefore singular.

6. Prove that the determinant of any upper triangular matrix $U$ or lower triangular matrix $L$ is simply the product of the diagonal elements: If

$$U_{i,j} = 0 \text{ for } 1 \leq j < i < n \text{ and } L_{i,j} = 0 \text{ for } 1 \leq i < j < n$$

then

$$\det(U) = \prod_{i=1}^{n} U_{i,i} \text{ and } \det(L) = \prod_{i=1}^{n} L_{i,i}$$

If $U$ is upper triangular, then we simply expand along the first column:

$$\det(U) = \sum_{i=1}^{n} (-1)^{i+1} U_{i,1} M_{i,1}.$$

But $U_{i,1} = 0$ for $i > 1$, so we get that

$$\det(U) = (-1)^{1+1} U_{1,1} M_{1,1} = U_{1,1} M_{1,1}.$$

Repeating this process inductively, notice that in computing the determinant $M_{1,1}$ we can expand along the first column, since it is also upper diagonal. Therefore, we end up with

$$\det(U) = U_{1,1} \left( U_{2,2} \hat{M}_{2,2} \right),$$

where $\hat{M}_{2,2}$ is the determinant of the matrix corresponding $U$ with its first two rows and columns removed. We do this $n$ times, we end up with

$$\det(U) = U_{1,1} U_{2,2} \cdots U_{n,n} = \prod_{i=1}^{n} U_{i,i}.$$

The same idea works for a lower triangular matrix $L$, except we always expand along the first row.

7. Suppose that a matrix $A$ can be written as $A = LU$, where $L$ is lower triangular and $U$ is upper triangular, specifically given

$$A = \begin{bmatrix} 1 & 0 & 0 \\ g & 1 & 0 \\ h & i & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

What is $\det(A)$? This process of decomposing a matrix into the product of an upper and lower triangular matrix is known as *LU factorization*.

Since $A = LU$, we get that $\det(A) = \det(LU)$, but $\det(LU) = \det(L)\det(U)$. Notice that $\det(L) = 1$ and $\det(U) = adf$, thus:

$$\begin{aligned}\det(A) &= \det(LU)\\ &= \det(L)\det(U)\\ &= 1\,(adf)\\ &= adf.\end{aligned}$$

8. Find values of $\lambda$ such that the following systems of equations have a nontrivial (nonzero) solution.

Clearly $x = y = 0$ is a solution to all of the systems, since each one is homogeneous. So we need to write each equation in matrix form and find values of $\lambda$ which make the determinant of the resulting matrix zero.

(a) $$\begin{aligned}(6-\lambda)x - 4y &= 0\\ -2x + (4-\lambda)y &= 0\end{aligned} \quad\longrightarrow\quad \begin{bmatrix} 6-\lambda & -4 \\ -2 & 4-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned}\det\left(\begin{bmatrix} 6-\lambda & -4 \\ -2 & 4-\lambda \end{bmatrix}\right) &= (6-\lambda)(4-\lambda) - 8\\ &= \lambda^2 - 10\lambda + 16\\ &= (\lambda-2)(\lambda-8)\end{aligned}$$

Setting $(\lambda-2)(\lambda-8) = 0$ gives $\lambda = 2, 8$.

(b) $$\begin{aligned}(-3-\lambda)x + 5y &= 0\\ 7x + (-1-\lambda)y &= 0\end{aligned} \quad\longrightarrow\quad \begin{bmatrix} -3-\lambda & 5 \\ 7 & -1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned}\det\left(\begin{bmatrix} -3-\lambda & 5 \\ 7 & -1-\lambda \end{bmatrix}\right) &= (-3-\lambda)(-1-\lambda) - 35\\ &= \lambda^2 + 4\lambda - 32\\ &= (\lambda-4)(\lambda+8)\end{aligned}$$

Setting $(\lambda-4)(\lambda+8) = 0$ gives $\lambda = 4, -8$.

(c) $$\begin{aligned}(12-\lambda)x + y &= 0\\ -6x + (5-\lambda)y &= 0\end{aligned} \quad\longrightarrow\quad \begin{bmatrix} 12-\lambda & 1 \\ -6 & 5-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned}\det\left(\begin{bmatrix} 12-\lambda & 1 \\ -6 & 5-\lambda \end{bmatrix}\right) &= (12-\lambda)(5-\lambda) + 6\\ &= \lambda^2 - 17\lambda + 66\\ &= (\lambda-6)(\lambda-11)\end{aligned}$$

Setting $(\lambda-6)(\lambda-11)=0$ gives $\lambda=6,11$.

(d) $\begin{array}{c}(2-\lambda)x+8y=0\\ 6x+(4-\lambda)y=0\end{array} \quad \longrightarrow \quad \begin{bmatrix} 2-\lambda & 8 \\ 6 & 4-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{aligned}\det\left(\begin{bmatrix} 2-\lambda & 8 \\ 6 & 4-\lambda \end{bmatrix}\right) &= (2-\lambda)(4-\lambda)-48 \\ &= \lambda^2-6\lambda-40 \\ &= (\lambda-10)(\lambda+4)\end{aligned}$$

Setting $(\lambda-10)(\lambda+4)=0$ gives $\lambda=10,-4$.

9. For each system from problem 8, find the corresponding nontrivial solutions for each value of $\lambda$ found.

(a) For $\lambda=2$ we have the system $\begin{array}{c}4x-4y=0\\ -2x+2y=0\end{array}$. Notice that the first equation is just twice the second, therefore, the solution is simply all points which satisfy one of the equations, so we choose the first: $4x-4y=0$, or $x=y$. Thus, the solution is the set of all points of the form $(x,x)$ or $(y,y)$.

For $\lambda=8$, we have the system $\begin{array}{c}-2x-4y=0\\ -2x-4y=0\end{array}$. Notice that both of these equations are exactly the same, with solution $x=-2y$. Thus the solution is the set of all points of the form $(-2y,y)$ or $\left(x,-\frac{1}{2}x\right)$.

(b) For $\lambda=4$ we have the system $\begin{array}{c}-7x+5y=0\\ 7x-5y=0\end{array}$. Notice that the first equation is just minus the second, therefore, the solution is simply all points which satisfy one of the equations, so we choose the first: $-7x+5y=0$, or $y=\frac{7}{5}x$. Thus, the solution is the set of all points of the form $(x,\frac{7}{5}x)$ or $(\frac{5}{7}y,y)$.

For $\lambda=-8$, we have the system $\begin{array}{c}5x+5y=0\\ 7x+7y=0\end{array}$. Notice that both of these equations are exactly the same: $x+y=0$, after dividing through by 5 and 7, respectively. The solution is $x=-y$, and therefore the points which satisfy the equation are of the form $(x,-x)$ or $(-y,y)$.

(c) For $\lambda=6$ we have the system $\begin{array}{c}6x+y=0\\ -6x-y=0\end{array}$. Notice that the first equation is just minus the second, therefore, the solution is simply all points which satisfy one of the equations, so we choose the first: $6x+y=0$, or $y=-6x$. Thus, the solution is the set of all points of the form $(x,-6x)$ or $(-\frac{1}{6}y,y)$.

For $\lambda = 11$, we have the system $\begin{array}{r} x + y = 0 \\ -6x - 6y = 0 \end{array}$. Notice that both of these equations are exactly the same: $x + y = 0$, after dividing through by 5 and 7, respectively. The solution is $x = -y$, and therefore the points which satisfy the equation are of the form $(x, -x)$ or $(-y, y)$.

(d) For $\lambda = 10$ we have the system $\begin{array}{r} -8x + 8y = 0 \\ 6x - 6y = 0 \end{array}$. Yet again, these equations are multiples of each other, both of the form $x - y = 0$. The solution is the set of all points of the form $(x, x)$ or $(y, y)$.

For $\lambda = -4$, we have the system $\begin{array}{r} 6x + 8y = 0 \\ 6x + 8y = 0 \end{array}$. Notice that both of these equations are exactly the same. The solution is $x = -\frac{4}{3}y$ or $y = -\frac{3}{4}x$. The points which satisfy the equation are $(x, -\frac{3}{4}x)$ or $(-\frac{4}{3}y, y)$.

10. Let $c$ be a scalar and $A$ be a $k \times l$ matrix. Explain why $cA = D_c A$, where $D_c$ is the $k \times k$ diagonal matrix with all $c$'s on its diagonal.

The following string of inequalities shows how this is true:

$$\begin{aligned} A &= I_k A \\ cA &= cI_k A \\ cA &= D_c A. \end{aligned}$$

11. Use your argument from problem 10 to show that $\det(cA) = c^n \det(A)$ if $A$ is any $n \times n$ matrix and $c$ is any scalar.

Since we have $cA = D_c A$, then

$$\begin{aligned} \det(cA) &= \det(D_c A) \\ &= \det(D_c)\det(A) \\ &= c^n \det(A). \end{aligned}$$

12. Show that $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

If $A$ is an $n \times n$ invertible matrix, then $A^{-1}A = I_n$, thus $\det\left(A^{-1}A\right) = \det(I_n)$, but $\det(I_n) = 1$, therefore, we have

$$\begin{aligned} \det\left(A^{-1}A\right) &= 1 \\ \det\left(A^{-1}\right)\det(A) &= 1. \end{aligned}$$

Solving the last line of the above string of equations for $\det(A^{-1})$ gives $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

13. Explain why $\det(A^T) = \det(A)$.

This must be true since determinants can be taken along any row or column.

14. Let $A$ be an $n \times n$ matrix with $\det(A) \neq 0$.

(a) Find a formula for $\det(C)$ in terms of $n$ and $\det(A)$, where $C$ is the cofactor matrix of $A$.

If we start with equation (5.10) and then multiply both sides by $\det(A)$, we get

$$\det(A)A^{-1} = C^T.$$

Taking the determinant of both sides (and using the fact that $\det(C^T) = \det(C)$) gives

$$\begin{aligned}
\det\left(\det(A)A^{-1}\right) &= \det(C) \\
\det\left(\det(A)I_n A^{-1}\right) &= \det(C) \\
\det\left(\det(A)I_n\right)\det\left(A^{-1}\right) &= \det(C) \\
(\det(A))^n \, \frac{1}{\det(A)} &= \det(C) \\
(\det(A))^{n-1} &= \det(C).
\end{aligned}$$

Therefore, we can conclude that

$$\det(C) = (\det(A))^{n-1}.$$

(b) Find a formula for $C^{-1}$ in terms of $A$ and $n$ if $C$ is $A$'s cofactor matrix.

We start with the simple relation $C^T = \det(A)\,A^{-1}$ and perform some simple matrix manipulations:

$$\begin{aligned}
C^T &= \det(A)\,A^{-1} \\
\left(C^T\right)^T &= \left(\det(A)\,A^{-1}\right)^T \\
C &= \det(A)\,\left(A^{-1}\right)^T \\
C &= \det(A)\,\left(A^T\right)^{-1} \\
C\,C^{-1} &= \det(A)\,\left(A^T\right)^{-1}\,C^{-1} \\
I_n &= \det(A)\,\left(A^T\right)^{-1}\,C^{-1} \\
\frac{1}{\det(A)} I_n &= \left(A^T\right)^{-1}\,C^{-1}
\end{aligned}$$

$$\begin{aligned}
A^T \frac{1}{\det(A)} I_n &= A^T \left(A^T\right)^{-1} C^{-1} \\
A^T \frac{1}{\det(A)} I_n &= I_n \, C^{-1} \\
\frac{1}{\det(A)} \, A^T &= C^{-1}.
\end{aligned}$$

Therefore, we can conclude that

$$C^{-1} = \frac{1}{\det(A)} \, A^T.$$

15. Compute the determinants of each of the three types of $n \times n$ elementary matrices.

Section 5.3 has more details on the answer to this problem.

First we consider type I elementary matrices, which is the identity matrix with two rows swapped. Clearly if no rows were swapped, the determinant would be 1. Using the fact that when we expand along a row or column we must multiply each term by $(-1)^{i+j}$, it should be reasonable to think that the determinant of a type I elementary matrix should be -1. This is true since if we choose one of the rows that were swapped to expand along, we are off by a minus sign (as when we expand along the original row of the identity matrix).

Next, we consider type II elementary matrices, which consists of multiplying one row of the identity matrix by a scalar $k$. Notice that this matrix is still diagonal, and hence its determinant is the product of the diagonal entries, all of which are one except for the one with value $k$. Therefore, the determinant of a type II matrix is $k$, where $k$ is the scalar used to multiply one row of the matrix in question.

Finally, type III matrices: multiplying a row by a non-zero number $c$ and adding it to another row. Once again, a type III matrix is either upper triangular or lower triangular, with ones only on the diagonal. Hence, the determinant is equal to one and is independent of the scalar $c$.

16. Let $P(x_0, y_0)$ and $Q(x_1, y_1)$ be two distinct points of $\mathbb{R}^2$.

(a) Show that the line through these two points has the equation

$$\det\left(\begin{bmatrix} x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \\ x & y & 1 \end{bmatrix}\right) = 0.$$

Clearly, if we wish to fit the line points $P$ and $Q$ to the line $Ax+By+C=0$, we would end up with the first two rows of the matrix given above. If we assume that $x$ and $y$ are arbitrary points on the line, then we also get the last row of the above matrix. Taking a determinant gives

$$\det\left(\begin{bmatrix} x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \\ x & y & 1 \end{bmatrix}\right) = x_0\, y_1 - x_0\, y - x_1\, y_0 + x_1\, y + x\, y_0 - x\, y_1.$$

From algebra, we know that we can express the equation of a line as

$$y - y_0 = m(x - x_0),$$

where $m = \dfrac{y_1 - y_0}{x_1 - x_0}$. Thus, we have

$$\begin{aligned} y - y_0 &= \frac{y_1 - y_0}{x_1 - x_0}\,(x - x_0) \\ (y - y_0)(x_1 - x_0) &= (y_1 - y_0)(x - x_0) \\ y\, x_1 - y\, x_0 - y_0\, x_1 + y_0 x_0 &= y_1\, x - y_1\, x_0 - y_0\, x + y_0\, x_0 \end{aligned}$$

Notice that after canceling the $y_0\, x_0$ terms from both sides of the last equation above, we can move all the terms to the left hand side, and rearrange to get the answer in the order $x_0\, y_1 - x_0\, y - x_1\, y_0 + x_1\, y + x\, y_0 - x\, y_1$, which is the determinant given above.

(b) Use the formula in part (a) to find the equation of the line through the two points $P(-7, 4)$ and $Q(9, -5)$.

Using the formula from part a) gives:

$$\det\left(\begin{bmatrix} -7 & 4 & 1 \\ 9 & -5 & 1 \\ x & y & 1 \end{bmatrix}\right) = -1 + 16y + 9x,$$

Thus, the equation of the line is $9x + 16y - 1 = 0$.

17. Let $P(x_0, y_0, z_0)$, $Q(x_1, y_1, z_1)$, and $R(x_2, y_2, z_2)$ be three noncollinear points of $\mathbb{R}^3$.

(a) Show that the plane through these three points has the equation

$$\det\left(\begin{bmatrix} x_0 & y_0 & z_0 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x & y & z & 1 \end{bmatrix}\right) = 0$$

This problem is very similar to that of 16 a). The standard equation for a plane is $Ax + By + Cz + D = 0$. Three points uniquely define a plane, and the columns (from left to right) of the above matrix correspond to the constants $A$, $B$, $C$ and $D$. The fourth row corresponds to any other arbitrary point $(x, y, z)$ which lies on the plane.

$$\det\left(\begin{bmatrix} x_0 & y_0 & z_0 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x & y & z & 1 \end{bmatrix}\right) = \left[y_0\left(z_2 - z_1\right) + y_1\left(z_0 - z_2\right) + y_2\left(z_1 - z_0\right)\right] x$$

$$\begin{aligned} &+ \left[z_0\left(x_2 - x_1\right) + z_1\left(x_0 - x_2\right) + z_2\left(x_1 - x_0\right)\right] y \\ &+ \left[x_0\left(y_2 - y_1\right) + x_1\left(y_0 - y_2\right) + x_2\left(y_1 - y_0\right)\right] z \\ &+ x_0\left(y_1\, z_2 - y_2\, z_1\right) + x_1\left(y_2\, z_0 - y_0\, z_2\right) \\ &+ x_2\left(y_0\, z_1 - y_1\, z_0\right) \\ =\ & Ax + By + Cz + D. \end{aligned}$$

Setting $Ax + By + Cz + D = 0$ gives the desired result, thus, we arrive at the fact that the determinant formula given above is a plane passing through the three points $P$, $Q$ and $R$.

(b) Use the formula in part (a) to find the equation of the plane through the three points $P(-7, 4, 2)$, $Q(9, -5, 8)$, and $R(6, 11, -3)$.

Using the formula from part a) gives:

$$\det\left(\begin{bmatrix} -7 & 4 & 2 & 1 \\ 9 & -5 & 8 & 1 \\ 6 & 11 & -3 & 1 \\ x & y & z & 1 \end{bmatrix}\right) = 1069 - 229z - 158y - 3x$$

Thus, the equation of the plane is $-3x - 158y - 229z + 1069 = 0$.

18. Let $P(x_0, y_0)$, $Q(x_1, y_1)$, and $R(x_2, y_2)$ be three noncollinear points of $\mathbb{R}^2$.

(a) Show that the circle through these three points has the equation

$$\det\left(\begin{bmatrix} x_0^2 + y_0^2 & x_0 & y_0 & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x^2 + y^2 & x & y & 1 \end{bmatrix}\right) = 0$$

First, we realize that the equation of the circle is given by $(x - x_c)^2 + (y - y_c)^2 = r^2$, where $(x_c, y_c)$ is the center of the circle. Expanding this, we can express this as $A(x^2+y^2)+Bx+Cx+D=0$, for constants $A$, $B$, $C$ and $D$. Plugging in the points $(x_k, y_k)$, for $1 \le k \le 3$, into the equation of the circle gives the first three rows of the above matrix. Then assuming the fourth arbitrary point $(x, y)$ lies on the circle, we end up with the final row above.

(b) Use the formula in part (a) to find the equation of the circle through the three points $P(-7,4)$, $Q(9,-5)$, and $R(6,11)$.

Using the formula from part a) gives:

$$\det\left(\begin{bmatrix} 65 & -7 & 4 & 1 \\ 106 & 9 & -5 & 1 \\ 157 & 6 & 11 & 1 \\ x^2+y^2 & x & y & 1 \end{bmatrix}\right) = -229\left(x^2+\,y^2\right)+1115\,x+939\,y+18934.$$

Thus, the equation of the circle is $229\left(x^2+\,y^2\right)-1115\,x-939\,y-18934=0$.

19. Can you revise problem 18 in order to find a determinant equation for a general conic section passing through a certain number of points in the $xy$-plane? If yes, then test your formula on a set of points. If no, then explain why this is impossible.

If we wish to fit the points to a conic, then we start with the form $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$, pick 5 points $(x_k, y_k)$, $1 \le k \le 5$ and an arbitrary point $(x, y)$ to get the matrix determinant equation:

$$\det\left(\begin{bmatrix} x_0^2 & x_0\,y_0 & y_0^2 & x_0 & y_0 & 1 \\ x_1^2 & x_1\,y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2\,y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3\,y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4\,y_4 & y_4^2 & x_4 & y_4 & 1 \\ x^2 & x\,y & y^2 & x & y & 1 \end{bmatrix}\right) = 0.$$

Consider the set of points

$$\{(-1,2),(0,3),(1,1),(2,4),(-2,3)\},$$

and corresponding matrix equation

$$\det\left(\begin{bmatrix} 1 & -2 & 4 & -1 & 2 & 1 \\ 0 & 0 & 9 & 0 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 8 & 16 & 2 & 4 & 1 \\ 4 & -6 & 9 & -2 & 3 & 1 \\ x^2 & x\,y & y^2 & x & y & 1 \end{bmatrix}\right) = 0.$$

Computing this determinant and setting it to zero gives the conic

$$28\,x^2 - 8\,x\,y - 86\,y^2 + 80\,x + 394\,y - 408 = 0.$$

We graph both the points and the conic together in the figure below.

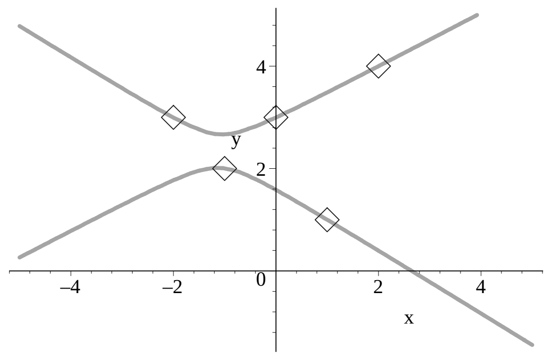


Figure 5.1: The five points and the conic section which passes through them all.

20. Can you revise problem 18 in order to find a determinant equation for a sphere passing through a certain number of points in space? If yes, then test your formula on a set of points. If no, then explain why this is impossible.

A sphere centered at the origin, with radius $r$ is given by the equation

$$x^2 + y^2 + z^2 = r^2.$$

However, if this sphere has a center at $(x_c, y_c, z_z)$ instead, the equation becomes

$$(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 = r^2.$$

Expanding this out, we end up with an equation of the form

$$A(x^2 + y^2 + z^2) + Bx + Cy + Dz + E = 0.$$

21. Verify that any $2 \times 2$ or $3 \times 3$ matrix $A$ that has two identical rows (or columns) must have $\det(A) = 0$.

## 5.3 Determinants Found by Triangularizing Matrices

1. Compute the determinants of the following matrices.

(a) $$\det\left(\begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = (1)(-1)(1) = -1$$

(b) $$\det\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 9 & 1 & 0 & 0 \\ 3 & 6 & 1 & 0 \\ 12 & -1 & -1 & 1 \end{bmatrix}\right) = (1)(1)(1)(1) = 1$$

(c) $$\det\left(\begin{bmatrix} 1 & -13 & -5 & -3 \\ 0 & -4 & 3 & 1 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 0 & 9 \end{bmatrix}\right) = (1)(-4)(5)(9) = -180$$

(d) $$\det\left(\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 9 & 7 & 0 & 0 \\ -2 & 8 & 4 & 4 & 0 \\ 6 & -2 & -5 & 2 & 1 \end{bmatrix}\right) = (3)(1)(7)(4)(1) = 84$$

(e) $$\det\left(\begin{bmatrix} 3 & 3 & 0 & 0 \\ 1 & 5 & 0 & 0 \\ 0 & 0 & -7 & 3 \\ 0 & 0 & 4 & 8 \end{bmatrix}\right) = (15-3)(-56-12) = -816$$

$$\text{(f) det}\left(\begin{bmatrix} 3 & 8 & 0 & 0 & 0 \\ -2 & 7 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & -8 \\ 0 & 0 & 0 & 3 & 6 \end{bmatrix}\right) = (21+16)(1)(30+24) = 1998$$

2. Compute the determinants of the following matrices by converting each to upper triangular form using only type III elementary matrices.

$$\text{(a)}\quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{4}{9} & 1 \end{bmatrix}$$

$$E_3E_2E_1 \begin{bmatrix} 1 & -1 & 3 \\ 2 & 7 & 7 \\ -4 & 8 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 9 & 1 \\ 0 & 0 & \frac{113}{9} \end{bmatrix}$$

$$\begin{aligned} \det\left(\begin{bmatrix} 1 & -1 & 3 \\ 2 & 7 & 7 \\ -4 & 8 & 1 \end{bmatrix}\right) &= \det\left(\begin{bmatrix} 1 & -1 & 3 \\ 0 & 9 & 1 \\ 0 & 0 & \frac{113}{9} \end{bmatrix}\right) \\ &= (1)(9)\left(\frac{113}{9}\right) = 113 \end{aligned}$$

$$\text{(b) } E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -9 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -12 & 0 & 0 & 1 \end{bmatrix} \qquad E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{3}{14} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{13}{14} & 0 & 1 \end{bmatrix} \qquad E_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{3}{187} & 1 \end{bmatrix}$$

$$E_6E_5E_3E_2E_1 \begin{bmatrix} 1 & 1 & -2 & 3 \\ 9 & -5 & 7 & 8 \\ 3 & 6 & 2 & -3 \\ 12 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -14 & 25 & -19 \\ 0 & 0 & \frac{187}{14} & -\frac{225}{14} \\ 0 & 0 & 0 & -\frac{3294}{187} \end{bmatrix}$$

$$\begin{aligned} \det\left(\begin{bmatrix} 1 & 1 & -2 & 3 \\ 9 & -5 & 7 & 8 \\ 3 & 6 & 2 & -3 \\ 12 & -1 & -1 & 1 \end{bmatrix}\right) &= \det\left(\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -14 & 25 & -19 \\ 0 & 0 & \frac{187}{14} & -\frac{225}{14} \\ 0 & 0 & 0 & -\frac{3294}{187} \end{bmatrix}\right) \\ &= (1)(-14)\left(\frac{187}{14}\right)\left(-\frac{3294}{187}\right) \\ &= 3294 \end{aligned}$$

(c) $$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -8 & 0 & 0 & 1 \end{bmatrix} \qquad E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{9}{5} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{23}{5} & 0 & 1 \end{bmatrix} \qquad E_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{4608}{581} & 1 \end{bmatrix}$$

$$\begin{aligned} \det\left(\begin{bmatrix} 1 & 4 & 9 & 5 \\ 3 & 7 & -7 & 3 \\ 2 & -11 & 5 & -1 \\ 8 & 9 & 0 & 2 \end{bmatrix}\right) &= \det\left(\begin{bmatrix} 1 & 4 & 9 & 5 \\ 0 & -5 & -34 & -12 \\ 0 & 0 & \frac{581}{5} & \frac{173}{5} \\ 0 & 0 & 0 & -\frac{4608}{581} \end{bmatrix}\right) \\ &= (1)(-5)\left(\frac{581}{5}\right)\left(-\frac{4608}{581}\right) \\ &= 4608 \end{aligned}$$

3. Compute the determinants of the matrices from problem 2 by converting each to lower triangular form using only type III elementary matrices.

(a) $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & -\frac{25}{49} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$E_3E_2E_1 \begin{bmatrix} 1 & -1 & 3 \\ 2 & 7 & 7 \\ -4 & 8 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{113}{49} & 0 & 0 \\ 30 & -49 & 0 \\ -4 & 8 & 1 \end{bmatrix}$$

$$\begin{aligned} \det\left(\begin{bmatrix} 1 & -1 & 3 \\ 2 & 7 & 7 \\ -4 & 8 & 1 \end{bmatrix}\right) &= \det\left(\begin{bmatrix} -\frac{113}{49} & 0 & 0 \\ 30 & -49 & 0 \\ -4 & 8 & 1 \end{bmatrix}\right) \\ &= -\frac{113}{9}(-49)(1) = 113 \end{aligned}$$

(b) $E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 15 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_5 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E_6 = \begin{bmatrix} 1 & -\frac{7}{48} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_6E_5E_3E_2E_1 \begin{bmatrix} 1 & 1 & -2 & 3 \\ 9 & -5 & 7 & 8 \\ 3 & 6 & 2 & -3 \\ 12 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{549}{8} & 0 & 0 & 0 \\ 498 & 48 & 0 & 0 \\ 39 & 3 & -1 & 0 \\ 12 & -1 & -1 & 1 \end{bmatrix}$$

$$\begin{aligned} \det\left( \begin{bmatrix} 1 & 1 & -2 & 3 \\ 9 & -5 & 7 & 8 \\ 3 & 6 & 2 & -3 \\ 12 & -1 & -1 & 1 \end{bmatrix} \right) &= \det\left( \begin{bmatrix} -\frac{549}{8} & 0 & 0 & 0 \\ 498 & 48 & 0 & 0 \\ 39 & 3 & -1 & 0 \\ 12 & -1 & -1 & 1 \end{bmatrix} \right) \\ &= -\frac{549}{8}(48)(-1)(1) \\ &= 3294 \end{aligned}$$

(c) $$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{3}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{7}{5} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_5 = \begin{bmatrix} 1 & 0 & -\frac{9}{5} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E_6 = \begin{bmatrix} 1 & -\frac{17}{39} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \det\left(\begin{bmatrix} 1 & 4 & 9 & 5 \\ 3 & 7 & -7 & 3 \\ 2 & -11 & 5 & -1 \\ 8 & 9 & 0 & 2 \end{bmatrix}\right) &= \det\left(\begin{bmatrix} -\frac{384}{13} & 0 & 0 & 0 \\ -\frac{3}{5} & -\frac{78}{5} & 0 & 0 \\ 6 & -\frac{13}{2} & 5 & 0 \\ 8 & 9 & 0 & 2 \end{bmatrix}\right) \\ &= -\frac{384}{13}\left(-\frac{78}{5}\right)(5)(2) \\ &= 4608 \end{aligned}$$

4. Compute the determinants of the following matrices by converting each to upper triangular form:

(a) Note that there was one row swap in the above calculation.

$$\begin{aligned} \det\left(\begin{bmatrix} 2 & -1 & 3 & 1 \\ 4 & -2 & 5 & 5 \\ 3 & 7 & -1 & 4 \\ 6 & 2 & 2 & -8 \end{bmatrix}\right) &= -\det\left(\begin{bmatrix} 2 & -1 & 3 & 1 \\ 0 & \frac{17}{2} & -\frac{11}{2} & \frac{5}{2} \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & -\frac{404}{17} \end{bmatrix}\right) \\ &= -404 \end{aligned}$$

(b) Note that there were two row swaps in the above calculation.

$$\begin{aligned} \det\left(\begin{bmatrix} 1 & -2 & 0 & 4 \\ -1 & 2 & 1 & 3 \\ 2 & -4 & 0 & 5 \\ 5 & -3 & 2 & 1 \end{bmatrix}\right) &= \det\left(\begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 7 & 2 & -19 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & -3 \end{bmatrix}\right) \\ &= -21 \end{aligned}$$

5. Verify your answers to problem 4 by computing the determinant via the method of expanding along a row or column.

Answers will vary depending upon which row/column you expand along, but the answers will agree with those of problem 4.

6. Let $A$ and $B$ be two square matrices of any two sizes.

(a) Show that the square diagonal block matrix $C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ for appropriate size 0 matrices has $\det(C) = \det(A)\det(B)$.

This is obvious upon expanding along the first column (or row) of $C$.

(b) Explain why

$$C^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix}$$

As a consequence, what is the inverse of a diagonal matrix $D$?

First, let us consider the product:

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix}$$

If you perform the multiplication entry by entry in the $A$ square, or in the $B$ square, notice that you have the definition of the inverse to $A$, and $B$, respectively. Furthermore, if you select an entry off diagonal in the resulting matrix, it is zero due to the construction of the matrix $C$ itself.

As a result, the inverse to a diagonal matrix is itself a diagonal matrix, whose diagonal entries are the multiplicative reciprocals of the diagonal entries of $D$.

7. Let $A$ be a square $n \times n$ matrix. If $A$ can be lower/upper triangularized using only type III elementary row operations, then how many of these operations do you expect to need? Give you answer as simply as possible.

Since there are $n^2$ entries, and thus $\dfrac{n^2 - n}{2}$ entries above/below the diagonal, there will be at most $\dfrac{n^2 - n}{2}$ type III elementary row operations to reduce the matrix to lower/upper triangularize the matrix.

8. Give an example of each type of elementary matrix for size $3 \times 3$ and give their inverses.

$$E_I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, E_I^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_{II} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{II}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/k & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{III} = \begin{bmatrix} 1 & 0 & 0 \\ n & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{III}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -n & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

9. If $A$ is a product of elementary matrices of size $n \times n$, then is $A$ invertible or not, and why?

Yes, $A$ must be invertible, since each elementary matrix is invertible.

10. Let $A$ be a square matrix with $\det(A) \neq 0$. Can you find $A^{-1}$ by successively applying elementary matrices to $A$ in order to produce the $n \times n$ identity matrix $I_n$? If yes, explain how. If no, explain why not.

Indeed you can. By first making the matrix upper triangular, and then lower triangular, you create a diagonal matrix, whose inverse we know from problem 6.

11. For a square matrix $A$, explain why rref($A$) will be the identity matrix unless rref($A$) contains at least one row of all zeroes.

This follows directly from problem 10. Elementary row operations can be used to get the matrix to the identity if $A$ is invertible. If $A$ is not invertible, elementary row operations can be used to remove all rows which are linear combinations of other rows. This will yield at least one row of all zeros.

## 5.4 *LU* Factorization

1. Solve the following systems by forward or backward substitution:

(a) $\{y_1 = -1, y_2 = 0, y_3 = 4\}$

(b) $\left\{x_1 = -\frac{1}{16}, x_2 = -\frac{9}{8}, x_3 = 0\right\}$

(c) $\left\{x_1 = \frac{41}{2}, x_2 = \frac{327}{20}, x_3 = -\frac{9}{2}, x_4 = -\frac{1}{5}\right\}$

(d) $\{y_1 = 1, y_2 = 2, y_3 = -11, y_4 = 68\}$

2. Compute the LU factorization of the following matrices.

(a) $$\begin{bmatrix} 1 & 3 & 1 & -7 \\ -2 & 1 & -2 & 6 \\ 0 & 8 & 1 & -1 \\ -2 & 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & \frac{8}{7} & 1 & 0 \\ -2 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 & -7 \\ 0 & 7 & 0 & -8 \\ 0 & 0 & 1 & \frac{57}{7} \\ 0 & 0 & 0 & -\frac{163}{7} \end{bmatrix}$$

(b) $$\begin{bmatrix} 8 & -4 & -8 & -9 \\ 4 & 1 & 2 & 8 \\ -1 & 0 & 1 & -1 \\ 0 & -2 & 0 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ -\frac{1}{8} & -\frac{1}{6} & 1 & 0 \\ 0 & -\frac{2}{3} & 4 & 1 \end{bmatrix} \begin{bmatrix} 8 & -4 & -8 & -9 \\ 0 & 3 & 6 & \frac{25}{2} \\ 0 & 0 & 1 & -\frac{1}{24} \\ 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

(c) $$\begin{bmatrix} -1 & 0 & -3 & 5 \\ -2 & 2 & 7 & -9 \\ 3 & -5 & -2 & 10 \\ 6 & 8 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -3 & -\frac{5}{2} & 1 & 0 \\ -6 & 4 & -\frac{142}{43} & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & -3 & 5 \\ 0 & 2 & 13 & -19 \\ 0 & 0 & \frac{43}{2} & -\frac{45}{2} \\ 0 & 0 & 0 & \frac{1578}{43} \end{bmatrix}$$

(d) $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ -4 & -3 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ -4 & -3 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(e) $$\begin{bmatrix} 1 & -3 & 5 & 8 \\ -2 & 1 & -7 & -2 \\ 3 & 0 & 1 & -7 \\ 0 & -3 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -\frac{9}{5} & 1 & 0 \\ 0 & \frac{3}{5} & -\frac{16}{43} & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 5 & 8 \\ 0 & -5 & 3 & 14 \\ 0 & 0 & -\frac{43}{5} & -\frac{29}{5} \\ 0 & 0 & 0 & -\frac{411}{43} \end{bmatrix}$$

(f) $$\begin{bmatrix} 1 & -6 & 8 & -1 \\ 0 & 1 & -5 & -2 \\ 3 & 0 & 1 & -3 \\ -4 & -3 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 18 & 1 & 0 \\ -4 & -27 & -\frac{98}{67} & 1 \end{bmatrix} \begin{bmatrix} 1 & -6 & 8 & -1 \\ 0 & 1 & -5 & -2 \\ 0 & 0 & 67 & 36 \\ 0 & 0 & 0 & -\frac{291}{67} \end{bmatrix}$$

3. Compute the determinant of each of the matrices from problem 2.

(a) $\det\left(\begin{bmatrix} 1 & 3 & 1 & -7 \\ -2 & 1 & -2 & 6 \\ 0 & 8 & 1 & -1 \\ -2 & 1 & 0 & -1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 3 & 1 & -7 \\ 0 & 7 & 0 & -8 \\ 0 & 0 & 1 & \frac{57}{7} \\ 0 & 0 & 0 & -\frac{163}{7} \end{bmatrix}\right) = -163$

(b) $\det\left(\begin{bmatrix} 8 & -4 & -8 & -9 \\ 4 & 1 & 2 & 8 \\ -1 & 0 & 1 & -1 \\ 0 & -2 & 0 & -9 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 8 & -4 & -8 & -9 \\ 0 & 3 & 6 & \frac{25}{2} \\ 0 & 0 & 1 & -\frac{1}{24} \\ 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix}\right) = -12$

(c) $\det\left(\begin{bmatrix} -1 & 0 & -3 & 5 \\ -2 & 2 & 7 & -9 \\ 3 & -5 & -2 & 10 \\ 6 & 8 & -1 & 5 \end{bmatrix}\right) = \det\left(\begin{bmatrix} -1 & 0 & -3 & 5 \\ 0 & 2 & 13 & -19 \\ 0 & 0 & \frac{43}{2} & -\frac{45}{2} \\ 0 & 0 & 0 & \frac{1578}{43} \end{bmatrix}\right) = -1578$

(d) $\det\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ -4 & -3 & 5 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 1$

(e) $\det\left(\begin{bmatrix} 1 & -3 & 5 & 8 \\ -2 & 1 & -7 & -2 \\ 3 & 0 & 1 & -7 \\ 0 & -3 & 5 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & -3 & 5 & 8 \\ 0 & -5 & 3 & 14 \\ 0 & 0 & -\frac{43}{5} & -\frac{29}{5} \\ 0 & 0 & 0 & -\frac{411}{43} \end{bmatrix}\right) = -411$

(f) $\det\left(\begin{bmatrix} 1 & -6 & 8 & -1 \\ 0 & 1 & -5 & -2 \\ 3 & 0 & 1 & -3 \\ -4 & -3 & 5 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & -6 & 8 & -1 \\ 0 & 1 & -5 & -2 \\ 0 & 0 & 67 & 36 \\ 0 & 0 & 0 & -\frac{291}{67} \end{bmatrix}\right) = -291$

4. Solve the following systems of equations by $LU$ factorization, performing forward and backward substitution.

(a) $\{x_1 = -34, x_2 = 49, x_3 = 12, x_4 = 0\}$

(b) $\{x_1 = 41, x_2 = 14, x_3 = -3, x_4 = 9\}$

(c) $\{x_1 = -39, x_2 = 75, x_3 = -67, x_4 = 7\}$

## 5.5 Inverses from *rref*

1. Use the method of row reducing $(A|I_n)$ to compute the inverse to each of the following matrices:

(a) $$\begin{bmatrix} 1 & -1 & 2 \\ -1 & 3 & -1 \\ 1 & 2 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{4}{11} & -\frac{2}{11} & \frac{5}{11} \\ \frac{3}{11} & \frac{4}{11} & \frac{1}{11} \\ \frac{5}{11} & \frac{3}{11} & -\frac{2}{11} \end{bmatrix}$$

(b) $$\begin{bmatrix} 3 & -4 & 1 \\ 3 & 3 & -1 \\ 1 & 2 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{4}{29} & \frac{6}{29} & -\frac{1}{29} \\ -\frac{5}{29} & \frac{7}{29} & -\frac{6}{29} \\ -\frac{3}{29} & \frac{10}{29} & -\frac{21}{29} \end{bmatrix}$$

(c) $$\begin{bmatrix} 2 & 3 & 1 \\ -1 & -5 & 2 \\ 1 & -7 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{8} & -\frac{13}{32} & \frac{11}{32} \\ \frac{1}{8} & \frac{3}{32} & -\frac{5}{32} \\ \frac{3}{8} & \frac{17}{32} & -\frac{7}{32} \end{bmatrix}$$

(d) $$\begin{bmatrix} -1 & 2 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 2 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

(e) $$\begin{bmatrix} 1 & 3 & -5 \\ 1 & 4 & 6 \\ 2 & 3 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{5} & -\frac{18}{25} & \frac{19}{25} \\ \frac{1}{10} & \frac{17}{50} & -\frac{11}{50} \\ -\frac{1}{10} & \frac{3}{50} & \frac{1}{50} \end{bmatrix}$$

(f) $$\begin{bmatrix} -1 & 2 & 0 \\ 5 & 0 & 2 \\ -4 & 2 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & -\frac{1}{3} \\ \frac{2}{3} & 0 & -\frac{1}{6} \\ -\frac{5}{6} & \frac{1}{2} & \frac{5}{6} \end{bmatrix}$$

(g) $$\begin{bmatrix} -1 & 2 & 1 & 0 \\ -2 & 2 & 1 & 1 \\ -3 & -1 & 2 & 1 \\ 2 & -5 & 6 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{4}{45} & \frac{4}{15} & -\frac{17}{45} & \frac{1}{9} \\ \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} & 0 \\ \frac{19}{45} & -\frac{2}{15} & \frac{1}{45} & \frac{1}{9} \\ -\frac{53}{45} & \frac{19}{15} & -\frac{17}{45} & \frac{1}{9} \end{bmatrix}$$

(h) $$\begin{bmatrix} 1 & 3 & -5 & 2 \\ 1 & 4 & 6 & 0 \\ 2 & 3 & 7 & 1 \\ -3 & 6 & -4 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{41}{147} & \frac{3}{49} & \frac{5}{147} & -\frac{29}{147} \\ \frac{29}{294} & \frac{16}{49} & -\frac{61}{294} & \frac{1}{294} \\ -\frac{11}{98} & -\frac{3}{49} & \frac{13}{98} & \frac{3}{98} \\ -\frac{10}{147} & -\frac{33}{49} & \frac{92}{147} & \frac{25}{147} \end{bmatrix}$$

(i) $$\begin{bmatrix} -1 & 2 & 0 & 1 \\ 3 & 0 & 2 & 2 \\ -4 & 2 & 1 & -3 \\ 0 & 2 & 0 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{16}{17} & \frac{3}{17} & -\frac{6}{17} & -\frac{10}{17} \\ \frac{22}{17} & \frac{2}{17} & -\frac{4}{17} & -\frac{19}{34} \\ -\frac{13}{17} & \frac{5}{17} & \frac{7}{17} & \frac{6}{17} \\ -\frac{11}{17} & -\frac{1}{17} & \frac{2}{17} & \frac{9}{17} \end{bmatrix}$$

2. In this section, it was shown that row reducing the matrix $(A|I_2)$ resulted in the correct value of $A^{-1}$. The first step in this process required that $a \neq 0$. Repeat this procedure, but this time assume that $a = 0$. You may assume that $b \neq 0$ and $c \neq 0$. Remember that you cannot swap rows.

The simplest thing to do would be to add $\frac{1}{c}$ times row two to row one, and then proceed similar to the book.

3. If $A$ is square but has no inverse, then what does rref$(A|I_n)$ produce and how does it tell us that $A$ has no inverse? Give some examples.

You will not be able to reduce to rref$(A|I_n)$ to rref$\left(I_n|A^{-1}\right)$, as the left square matrix will not be $I_n$. As examples:

$$\text{rref}\left(\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix}$$
$$\text{rref}\left(\begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ -1 & -2 & -1 & 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

4. Explain why if $E$ is an elementary matrix, $E^{-1}$ is also.

Clearly, each elementery row operation is reversible, and as a result, the matrix representation of each elementary row operation must be invertible. For instance, see problem 8 from the previous section.

5. Explain why for any square matrix $A$ that is invertible, $A^{-1}$ is a product of elementary matrices.

Assume $A$ is invertible, by problem 10 of the previous section we know that $A$ can be transformed into the identity matrix through multiplication by elementary matrices $E_1$ through $E_n$. E.g.

$$E_n E_{n-1} \cdots E_2 E_1 A = I,$$

Notice that from the above formulation $A^{-1} = E_n E_{n-1} \cdots E_2 E_1$, hence $A^{-1}$ is a product of elementary matrices.

6. Explain why for any square matrix $A$ that is invertible, $A$ is a product of elementary matrices.

Under the same assumptions as the previous problem, we have

$$E_n E_{n-1} \cdots E_2 E_1 A = I,$$

which gives (after solving for $A$):

$$A = E_1^{-1} E_2^{-1} \cdots E_{n-1}^{-1} E_n^{-1}$$

Thus $A$ is a product of elementary matrices.

7. For $A$, the matrix in problem 1 part (c), write out both $A$ and $A^{-1}$ as products of elementary matrices.

If we start with $A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & -5 & 2 \\ 1 & -7 & 2 \end{bmatrix}$, then we get

$$E_9 E_8 \cdots E_2 E_1 A = I_3,$$

where

$$E_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{2}{7} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{17}{2} & 1 \end{bmatrix} \quad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{7}{32} \end{bmatrix}$$

$$E_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{5}{7} \\ 0 & 0 & 1 \end{bmatrix} \quad E_8 = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_9 = \begin{bmatrix} 1 & -\frac{3}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice that there are many different orders in which one could get the identity matrix, we have chosen one path. Using these matrices, we have

$$A^{-1} = E_9 E_8 \cdots E_2 E_1, \quad A = E_1^{-1} E_2^{-1} \cdots E_8^{-1} E_9^{-1},$$

where the inverse to each elementary matrix (see problem 8 from section 5.3) is given below:

$$E_1^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{7}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_5^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{17}{2} & 1 \end{bmatrix} \quad E_6^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{32}{7} \end{bmatrix}$$

$$E_7^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{5}{7} \\ 0 & 0 & 1 \end{bmatrix} \quad E_8^{-1} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_9^{-1} = \begin{bmatrix} 1 & \frac{3}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8. Is Theorem 5.5.1 still true if we replace "for all $B$" with "for at least one $B$"?

No. Using the properties of the biconditional, the following statement should be true:

If there exists a unique solution to the system $AX = B$ for at least one $B$, then $\det(A) \neq 0$.

However, if we let $B$ be the zero column matrix, then $X$ also being the zero column matrix will work regardless of $A$, even if $A$ is not invertible and thusly $\det(A) = 0$.

## 5.6 Cramer's Rule

1. Solve the following systems of equations by using Cramer's rule:

(a) $\left\{x_1 = \frac{5}{41}, x_2 = \frac{349}{82}, x_3 = \frac{19}{82}\right\}$

(b) $\left\{x_1 = \frac{7}{34}, x_2 = -\frac{21}{34}, x_3 = \frac{3}{17}\right\}$

(c) $\left\{x_1 = \frac{21}{4}, x_2 = -\frac{9}{4}, x_3 = \frac{11}{4}, x_4 = -\frac{11}{4}\right\}$

(d) $\left\{x_1 = \frac{11}{27}, x_2 = \frac{8}{27}, x_3 = -\frac{1}{27}, x_4 = \frac{2}{9}\right\}$

2. Given the following matrices,

$$A = \begin{bmatrix} 2 & -1 & 2 \\ 3 & 1 & -1 \\ 1 & 5 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 3 \\ 7 \end{bmatrix}$$

define $A_{i,B}$ to be as specified in this section, while $(A_i|B)$ to be the matrix found after removing column $i$ from $A$ and then augmenting the resulting matrix with $B$. Compute the determinants of the following matrices.

(a) $\det(A) = 59$ (b) $\det(A_{1,B}) = 26$ (c) $\det(A_{2,B}) = 87$

(d) $\det(A_{3,B}) = -12$ (e) $\det((A_1|B)) = 26$ (f) $\det((A_2|B)) = -87$

(g) $\det((A_3|B)) = -12$

3. Determine the relationship between $\det(A_{i,B})$ and $\det\left((A_i|B)\right)$, and use it to verify your results from problem 2.

Since each matrix of the form $A_{i,B}$ is the same as $(A_i|B)$ with just column swaps (think elementary matrix of type I with the transposed matrices), then either they will have the same value or will differ by a sign. If they differ by a sign, then there are an odd number of columns swaps required to transform one matrix to the other, and if they are of the same sign, an even number of column swaps are required.

4. Let $a$ be a scalar, in the matrix equation

$$\begin{bmatrix} a & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

For what values of $a$ does this system have one solution? For what values of $a$ does this system have no solution? For what values of $a$ does this system have infinitely many solutions?

Notice that $\det(A) = 3a - 2$, and thus when $a = \frac{2}{3}$, $A$ is not invertible and there is no solution to the system since the two resulting equations are not scalar multiples of each other (in which case there would be an infinite number of solutions for the specified value of $a$). For all other $a$, the matrix is invertible and thus only one solution will exist to the system for $a \neq \frac{2}{3}$.

5. Let $a$ be a scalar, in the matrix equation

$$\begin{bmatrix} 4 & 1 & 2 \\ a & 3 & a \\ 0 & -1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

For what values of $a$ does this system have one solution? For what values of $a$ does this system have no solution? For what values of $a$ does this system have infinitely many solutions?

Similar to the previous problem, we have $\det(a) = 60 - 3a$ and thus when $a = 20$, $\det(A) = 0$ and there are no solutions. For all other values of $A$, $\det(A) \neq 0$ and a unique (one) solution exists.

6. For the matrices $A = \begin{bmatrix} 3-i & 1+2i \\ 5+i & 1-i \end{bmatrix}$ and $B = \begin{bmatrix} 3+5i \\ 2-6i \end{bmatrix}$, use Cramer's Rule to solve the system $AX = B$.

$$X = \begin{bmatrix} -\frac{27}{113} - \frac{47}{113}i \\ \frac{365}{113} - \frac{51}{113}i \end{bmatrix}$$

# Chapter 6

# Basic Linear Algebra Topics

## 6.1 Vectors

1. If $\overrightarrow{u} = \langle 1, 3\rangle$, $\overrightarrow{v} = \langle 5, -7\rangle$, and $\overrightarrow{w} = \langle -4, 5\rangle$, perform the following operations:

(a) $\overrightarrow{u} - \overrightarrow{v} = \langle -4, 10\rangle$

(b) $4\overrightarrow{u} + 3\overrightarrow{w} = \langle -8, 27\rangle$

(c) $-6\overrightarrow{v} - 2\overrightarrow{w} = \langle -22, 32\rangle$

(d) $\overrightarrow{u} + \overrightarrow{v} - \overrightarrow{w} = \langle 10, -9\rangle$

(e) $2\overrightarrow{u} - 3\overrightarrow{v} + 6\overrightarrow{w} = \langle -37, 57\rangle$

(f) $-3\overrightarrow{w} + 2\overrightarrow{v} - 7\overrightarrow{u} = \langle 15, -50\rangle$

(g) $\alpha\overrightarrow{u} - \beta\overrightarrow{v} = \langle \alpha - 5\beta, 3\alpha + 7\beta\rangle$

(h) $\alpha\overrightarrow{u} - \beta\overrightarrow{v} + \gamma\overrightarrow{w} = \langle \alpha - 5\beta - 4\gamma, 3\alpha + 7\beta + 5\gamma\rangle$

(i) $\alpha\overrightarrow{w} + \beta\overrightarrow{v} = \langle -4\alpha + 5\beta, 5\alpha - 7\beta\rangle$

2. If $\overrightarrow{u} = \langle -1, 3, 2\rangle$, $\overrightarrow{v} = \langle 5, 0, -7\rangle$, and $\overrightarrow{w} = \langle -4, -2, 2\rangle$, perform the following operations:

(a) $3\overrightarrow{u} + 2\overrightarrow{v} = \langle 7, 9, -8\rangle$

(b) $6\overrightarrow{v} - 2\overrightarrow{w} = \langle 38, 4, -46\rangle$

(c) $7\overrightarrow{v} + 3\overrightarrow{w} = \langle 23, -6, -43\rangle$

(d) $2\overrightarrow{u} - \overrightarrow{v} + 3\overrightarrow{w} = \langle -19, 0, 17\rangle$

(e) $-5\overrightarrow{u} + 2\overrightarrow{w} - 3\overrightarrow{v} = \langle -18, -19, 15\rangle$

(f) $-6\overrightarrow{w} - \overrightarrow{v} + 8\overrightarrow{u} = \langle 11, 36, 11\rangle$

(g) $\alpha\overrightarrow{u} + \beta\overrightarrow{w} = \langle -\alpha - 4\beta, 3\alpha - 2\beta, 2\alpha + 2\beta\rangle$

(h) $\alpha\overrightarrow{u} - \beta\overrightarrow{v} + \gamma\overrightarrow{w} = \langle -\alpha - 5\beta - 4\gamma, 3\alpha - 2\gamma, 2\alpha + 7\beta + 2\gamma\rangle$

(i) $\alpha\overrightarrow{u} + \beta\overrightarrow{v} = \langle -\alpha + 5\beta, 3\alpha, 2\alpha - 7\beta\rangle$

3. If $\overrightarrow{u} = \langle -2, 3\rangle$ and $\overrightarrow{v} = \langle 1, 2\rangle$, find values of $\alpha$ and $\beta$ for each of the following vectors $\overrightarrow{w}$ so that $\overrightarrow{w} = \alpha\overrightarrow{u} + \beta\overrightarrow{v}$.

(a) $\langle 1, 3\rangle = \frac{1}{7}\overrightarrow{u} + \frac{9}{7}\overrightarrow{v}$ (b) $\langle -1, 2\rangle = \frac{4}{7}\overrightarrow{u} + \frac{1}{7}\overrightarrow{v}$

(c) $\langle -5, 6\rangle = \frac{16}{7}\overrightarrow{u} - \frac{3}{7}\overrightarrow{v}$ (d) $\langle -1, 5\rangle = 1\overrightarrow{u} + 1\overrightarrow{v}$

(e) $\langle 4, 4\rangle = -\frac{4}{7}\overrightarrow{u} + \frac{20}{7}\overrightarrow{v}$ (f) $\langle -8, 12\rangle = 4\overrightarrow{u} + 0\overrightarrow{v}$

4. If $\overrightarrow{u} = \langle -1, 1, 0\rangle$, $\overrightarrow{v} = \langle 1, 1, 0\rangle$, and $\overrightarrow{w} = \langle 0, 0, 1\rangle$, find values of $\alpha$, $\beta$, and $\gamma$ for each of the following vectors $\overrightarrow{x}$, so that $\overrightarrow{x} = \alpha\overrightarrow{u} + \beta\overrightarrow{v} + \gamma\overrightarrow{w}$:

(a) $\langle 1, 3, -1\rangle = 1\overrightarrow{u} + 2\overrightarrow{v} - 1\overrightarrow{w}$

(b) $\langle -1, 2, 4\rangle = \frac{3}{2}\overrightarrow{u} + \frac{1}{2}\overrightarrow{v} + 4\overrightarrow{w}$

(c) $\langle -5, 6, -4\rangle = \frac{11}{2}\overrightarrow{u} + \frac{1}{2}\overrightarrow{v} - 4\overrightarrow{w}$

5. If $\overrightarrow{u} = \langle -1, 1, 1\rangle$ and $\overrightarrow{v} = \langle 1, 0, 2\rangle$, find values of $\alpha$ and $\beta$ (if possible) for each of the following vectors $\overrightarrow{w}$ so that $\overrightarrow{w} = \alpha\overrightarrow{u} + \beta\overrightarrow{v}$:

| | | | |
|---|---|---|---|
| (a) | not possible | (b) | $\langle 1, 4, 14\rangle = 4\overrightarrow{u} + 5\overrightarrow{v}$ |
| (c) | not possible | (d) | not possible |
| (e) | not possible | (f) | $\langle -1, 2, 4\rangle = 2\overrightarrow{u} + 1\overrightarrow{v}$ |

6. For each of the vectors $\overrightarrow{w}$ of problem 5, construct a matrix $A \in \mathbb{R}^{3\times 3}$ whose columns are the vectors $\overrightarrow{u}$, $\overrightarrow{v}$, and $\overrightarrow{w}$, then compute $\det(A)$.

(a) $\det\left(\begin{bmatrix} -1 & 1 & -5 \\ 1 & 0 & 2 \\ 1 & 2 & 4 \end{bmatrix}\right) = -8$

(b) $\det\left(\begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 2 & 14 \end{bmatrix}\right) = 0$

(c) $\det\left(\begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}\right) = -1$

(d) $\det\left(\begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix}\right) = 1$

(e) $\det\left(\begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 2 & 1 \end{bmatrix}\right) = 13$

(f) $\det\left(\begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 2 \\ 1 & 2 & 4 \end{bmatrix}\right) = 0$

7. Interpret your results from problem 6.

If the determinant of the matrix is zero, then a solution to the equation in question should indeed exist. If the determinant is non-zero, no solution will exist.

8. Let $\overrightarrow{v} = \langle a, b\rangle \in \mathbb{R}^2$. The *slope* of $\overrightarrow{v}$ is $m_{\overrightarrow{v}} = \dfrac{b}{a}$ if $a \neq 0$ and otherwise it does not exist.

(a) Explain why $\overrightarrow{v} = \langle a, b\rangle$ and $\overrightarrow{w} = \langle -b, a\rangle$ are perpendicular (or orthogonal) vectors.

This is true simply because of the fact that $m_{\overrightarrow{v}} = -\dfrac{1}{m_{\overrightarrow{w}}}$.

(b) Let $\overrightarrow{w} \in \mathbb{R}^2$. Explain why $\overrightarrow{v}$ and $\overrightarrow{w}$ are parallel exactly when $m_{\overrightarrow{v}} = m_{\overrightarrow{w}}$, or both slopes do not exist.

Since the slope is the ratio of the $y$-coordinate to the $x$-coordinate (think rise over run), if the slopes $m_{\overrightarrow{v}}$ and $m_{\overrightarrow{w}}$ are the same, then the ratios of the $y$-coordinate to the $x$-coordinate for both vectors are equal, and thus the vectors are scalar multiples of each other, which is the definition parallel vectors. If there is no slope, then the two vectors are strictly vertical, and are once again parallel.

9. Find and plot the vector $\overrightarrow{w}$ with the original vector $\overrightarrow{v}$:

(a) A unit vector $\overrightarrow{w}$ in the opposite direction to $\overrightarrow{v} = \langle -2, 5\rangle$.

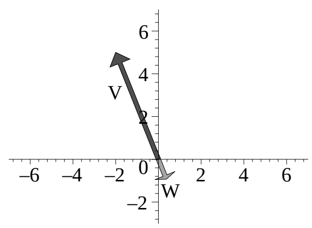


Figure 6.1: $\overrightarrow{w} = \langle \frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}\rangle$

(b) A vector $\overrightarrow{w}$ of length seven in the same direction as $\overrightarrow{v} = \langle 4, -7\rangle$.

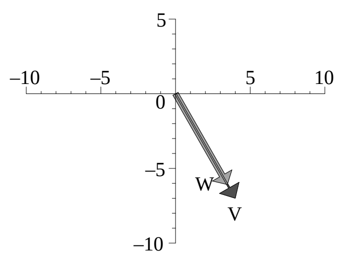


Figure 6.2: $\overrightarrow{w} = \langle \frac{28}{\sqrt{65}}, -\frac{49}{\sqrt{65}}\rangle$

(c) A vector $\overrightarrow{w}$ of length ten in the opposite direction to $\overrightarrow{v} = \langle 1, 6\rangle$.

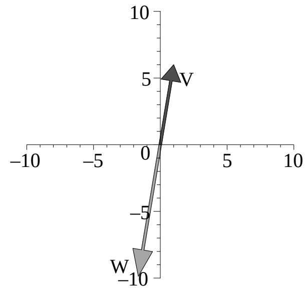


Figure 6.3: $\overrightarrow{w} = \langle -\frac{10}{\sqrt{37}}, -\frac{60}{\sqrt{37}}\rangle$

(d) A vector $\overrightarrow{w}$ of length three parallel to $\overrightarrow{v} = \langle -4, 8\rangle$ starting at $P(5, 8)$.

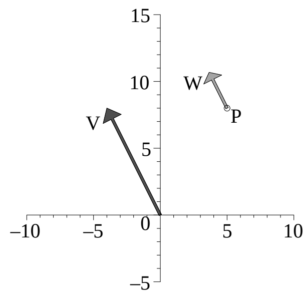


Figure 6.4: $\overrightarrow{w} = \langle -\frac{3}{\sqrt{5}}, \frac{6}{\sqrt{5}}\rangle$ at the base point $P(5, 8)$

(e) A vector $\overrightarrow{w}$ of length thirteen perpendicular to $\overrightarrow{v} = \langle -7, -3\rangle$.

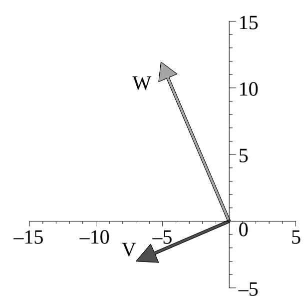


Figure 6.5: $\overrightarrow{w} = \langle -\frac{39}{\sqrt{58}}, \frac{91}{\sqrt{58}}\rangle$ is perpendicular to $\overrightarrow{v}$

(f) A vector $\overrightarrow{w}$ of length two perpendicular to $\overrightarrow{v} = \langle 4, -8\rangle$ starting at $P(3, 10)$.

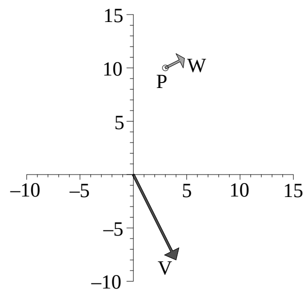


Figure 6.6: $\overrightarrow{w} = \langle \frac{4}{\sqrt{5}}, \frac{2}{\sqrt{5}}\rangle$ (at the base point $P(3, 10)$) is perpendicular to $\overrightarrow{v}$

(g) A vector $\overrightarrow{w}$ of length eight parallel to the line $4x + 7y = 10$.

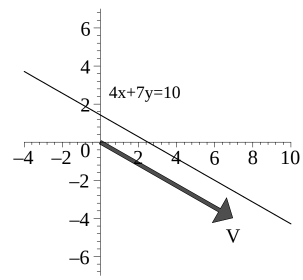


Figure 6.7: $\overrightarrow{w} = \langle \frac{56}{\sqrt{65}}, -\frac{32}{\sqrt{65}}\rangle$ is parallel to the line $4x + 7y = 10$.

10. Using trigonometry, find the angle between the two vectors $\overrightarrow{v} = \langle -2, 5\rangle$ and $\overrightarrow{w} = \langle 7, 3\rangle$.

Using the Law of Cosines, one has

$$85 = 29 + 58 - \sqrt{29}\,\sqrt{58}\cos(A),$$

where $A$ is the angle between the two vectors $\overrightarrow{v} = \langle -2, 5\rangle$ and $\overrightarrow{w} = \langle 7, 3\rangle$. Solving for $A$ gives

$$\begin{aligned} A &= \cos^{-1}\left(\frac{1}{29\sqrt{2}}\right) \\ &\approx 1.54641 \text{ radians} \\ &\approx 88.6028^\circ \end{aligned}$$

11. Let $\overrightarrow{v} = \langle -2, 5\rangle$ and $\overrightarrow{w} = \langle 7, 3\rangle$ be two adjacent sides of a triangle. Find the length of the third side of this triangle using the length of a vector.

The three points that define this triangle $(-2, 5)$, $(0, 0)$ and $(7, 3)$. This distance from $(-2, 5)$ to $(0, 0)$ is $|\overrightarrow{v}| = \sqrt{29}$, the distance from $(0, 0)$ to $(7, 3)$ is $|\overrightarrow{w}| = \sqrt{58}$, and lastly, the distance from $(-2, 5)$ to $(7, 3)$ is $|\overrightarrow{w} - \overrightarrow{v}| = \sqrt{85}$.

12. Find the distance between the two points $P(1, -4, 7, 0, 2)$, $Q(-9, 3, 5, -6, 8)$.

$$\begin{aligned} d(P,Q) &= \sqrt{(1-(-9))^2 + (-4-3)^2 + (7-5)^2 + (0-(-6))^2 + (2-8)^2} \\ &= \sqrt{225} \\ &= 15 \end{aligned}$$

## 6.2 Dot Product

1. Compute the dot products of the following pairs of vectors.

(a) $\langle -5, 2\rangle \cdot \langle 3, -2\rangle = -19$

(b) $\langle 1, 4\rangle \cdot \langle -6, 3\rangle = 6$

(c) $\langle -2-i, 1\rangle \cdot \langle 3, -2i\rangle = -6-i$

(d) $\langle 3, 2, -2\rangle \cdot \langle 4, 3, -2\rangle = 22$

(e) $\langle 1, 0, 4\rangle \cdot \langle 2, -6, 0\rangle = 2$

(f) $\langle 4, 2, 5\rangle \cdot \langle 1, 3, -2\rangle = 0$

2. Compute the norms of the following vectors.

(a) $|\langle -5, 2\rangle| = \sqrt{29}$

(b) $|\langle -3, 3\rangle| = 3\sqrt{2}$

(c) $|\langle 5, 3-2i\rangle| = \sqrt{38}$

(d) $|\langle 1, 0, -2\rangle| = \sqrt{5}$

(e) $\left|\left\langle \frac{1}{2}, 0, 4\right\rangle\right| = \frac{\sqrt{65}}{2}$

(f) $\left|\left\langle 3-6i, -\frac{1}{\sqrt{3}}, 2+i\right\rangle\right| = \frac{\sqrt{453}}{3}$

(g) $|\langle 1, 2, -1, 1\rangle| = \sqrt{7}$

(h) $|\langle -1, 2, 5, 3, 2\rangle| = \sqrt{43}$

(i) $|\langle 1-i, 8+2i, 3+2i, 1-i\rangle| = \sqrt{85}$

3. Determine the angle between the following pairs of vectors.

(a) $\langle -3,6\rangle$, $\langle 4,2\rangle \to \theta = \frac{\pi}{2}$

(b) $\langle 1,-4\rangle$, $\langle -2,2\rangle \to \theta = \pi - \cos^{-1}\left(\frac{5}{\sqrt{34}}\right) \approx 2.601173153$ radians

(c) $\langle -2,1\rangle$, $\langle 2,-1\rangle \to \theta = \pi$

(d) $\langle 1,0,-2\rangle$, $\langle 0,3,0\rangle \to \theta = \frac{\pi}{2}$

(e) $\langle 3,-2,4\rangle$, $\langle 2,-6,2\rangle \to \theta = \cos^{-1}\left(\frac{13}{\sqrt{319}}\right) \approx 0.7555999630$ radians

4. For each of the following vectors $\overrightarrow{v}$, find a second vector $\overrightarrow{w}$ such that $\overrightarrow{v} \perp \overrightarrow{w}$.

Answers can vary for this problem.

(a) $\langle 2,3\rangle \perp \langle -3,2\rangle$ (b) $\langle 5,-1\rangle \perp \langle 1,5\rangle$

(c) $\langle 1,0,5\rangle \perp \langle -5,2,1\rangle$ (d) $\langle 2,1,3\rangle \perp \langle 3,3,-3\rangle$

(e) $\langle 7,0,-2,2\rangle \perp \langle 0,0,3,-3\rangle$ (f) $\langle -2,3,1,5\rangle \perp \langle 2,1,1,0\rangle$

5. Prove the *Cauchy-Schwarz inequality*, which states that if $\overrightarrow{u}, \overrightarrow{v} \in \mathbb{R}^n$, then

$$|\overrightarrow{u} \cdot \overrightarrow{v}| \leq |\overrightarrow{u}|\,|\overrightarrow{v}|$$

First, we note that taking the absolute value of each side of equation 6.6 gives

$$|\overrightarrow{u} \cdot \overrightarrow{v}| = |\overrightarrow{u}|\,|\overrightarrow{v}|\,|\cos(\theta)|$$

Under the assumption that $\overrightarrow{u}$ and $\overrightarrow{v}$ are not both zero (in which case we have $0 \leq 0$) and using the fact that $0 \leq |\cos(\theta)| \leq 1$, we arrive at the Cauchy-Schwarz inequality.

6. Use the *Cauchy-Schwarz inequality* to prove the *Triangle inequality*, which states that for $\overrightarrow{u}, \overrightarrow{v} \in \mathbb{R}^n$:

$$|\overrightarrow{u} + \overrightarrow{v}| \leq |\overrightarrow{u}| + |\overrightarrow{v}|$$

*Hint: Start with the fact that* $|\overrightarrow{u} + \overrightarrow{v}|^2 = (\overrightarrow{u} + \overrightarrow{v}) \cdot (\overrightarrow{u} + \overrightarrow{v})$*, and expand the righthand side using the distributivity of the dot product.*

Since we are dealing with positive values on both sides of the inequality, and using the hint, we will prove the squared version instead:

$$|\overrightarrow{u} + \overrightarrow{v}|^2 \leq (|\overrightarrow{u}| + |\overrightarrow{v}|)^2$$

The hint tells us to use

$$|\overrightarrow{u}+\overrightarrow{v}|^2 = (\overrightarrow{u}+\overrightarrow{v})\cdot(\overrightarrow{u}+\overrightarrow{v})$$

and we now expand the righthand side and manipulate (here we also make use of the fact that these are real vectors):

$$\begin{aligned}(\overrightarrow{u}+\overrightarrow{v})\cdot(\overrightarrow{u}+\overrightarrow{v}) &= \overrightarrow{u}\cdot\overrightarrow{u}+\overrightarrow{u}\cdot\overrightarrow{v}+\overrightarrow{v}\cdot\overrightarrow{u}+\overrightarrow{v}\cdot\overrightarrow{v}\\ &= \overrightarrow{u}\cdot\overrightarrow{u}+2\overrightarrow{u}\cdot\overrightarrow{v}+\overrightarrow{v}\cdot\overrightarrow{v}\\ &= |\overrightarrow{u}|^2+|\overrightarrow{v}|^2+2\overrightarrow{u}\cdot\overrightarrow{v}\\ &\le |\overrightarrow{u}|^2+|\overrightarrow{v}|^2+2\,|\overrightarrow{u}\cdot\overrightarrow{v}|\\ &\le |\overrightarrow{u}|^2+|\overrightarrow{v}|^2+2\,|\overrightarrow{u}|\,|\overrightarrow{v}|\\ &= (|\overrightarrow{u}|+|\overrightarrow{v}|)^2 .\end{aligned}$$

We therefore now have

$$|\overrightarrow{u}+\overrightarrow{v}|^2 \le (|\overrightarrow{u}|+|\overrightarrow{v}|)^2,$$

and since the quantities under the squares are all positive, we arrive at

$$|\overrightarrow{u}+\overrightarrow{v}| \le |\overrightarrow{u}|+|\overrightarrow{v}|$$

7. Find the area and the interior angles of the triangle with the two adjacent sides $\overrightarrow{v} = \langle -5,-9\rangle$ and $\overrightarrow{w} = \langle 7,3\rangle$.

$$A = \frac{1}{2}\sqrt{106\cdot 58-(-62)} = 24$$

The angle between $\overrightarrow{v}$ and $\overrightarrow{w}$ is $\theta_1 \approx 2.482786618$ radians.

The angle between $\overrightarrow{v}$ and $\overrightarrow{w}-\overrightarrow{v}$ is $\theta_2 \approx 0.3805063781$ radians.

The angle between $\overrightarrow{w}$ and $\overrightarrow{v}-\overrightarrow{w}$ is $\theta_3 \approx 0.2782996610$ radians.

Notice that $\theta_1+\theta_2+\theta_3 = 3.141592657 \approx \pi$.

8. Find the area and the interior angles of the triangle with the two adjacent sides $\overrightarrow{v} = \langle -5,-9,12\rangle$ and $\overrightarrow{w} = \langle 7,3,-6\rangle$.

$$A = \frac{1}{2}\sqrt{94\cdot 250-17956} = 3\sqrt{154}$$

The angle between $\overrightarrow{v}$ and $\overrightarrow{w}$ is $\theta_1 \approx 2.634416612$ radians.

The angle between $\overrightarrow{v}$ and $\overrightarrow{v}-\overrightarrow{w}$ is $\theta_2 \approx 0.1915244633$ radians.

The angle between $\overrightarrow{w}$ and $\overrightarrow{w}-\overrightarrow{v}$ is $\theta_3 \approx 0.3156515777$ radians.

Notice that $\theta_1+\theta_2+\theta_3 = 3.141592653 \approx \pi$.

9. Consider the following two vectors in $\mathbb{C}^4$

$$\overrightarrow{v} = \langle -5+4i, -9-i, 12, 7i\rangle,\ \overrightarrow{w} = \langle 7-2i, 3, -6+5i, 1+i\rangle$$

For parts (c) and (d), you may wish to refer back to problems 5 and 6, and make use of the Triangle and Cauchy-Schwarz Inequalities.

(a) Compute $\overrightarrow{v}\cdot\overrightarrow{w}$ and $\overrightarrow{w}\cdot\overrightarrow{v}$.

$$\overrightarrow{v}\cdot\overrightarrow{w} = -135-38\,i,\ \ \overrightarrow{w}\cdot\overrightarrow{v} = -135+38\,i$$

(b) Find the norms of the two vectors $\overrightarrow{v}$ and $\overrightarrow{w}$.

$$|\overrightarrow{v}| = 2\sqrt{79},\ \ |\overrightarrow{w}| = 5\sqrt{5}$$

(c) Is it true that $|\overrightarrow{v}+\overrightarrow{w}| \leq |\overrightarrow{v}|+|\overrightarrow{w}|$?

The answer is yes:

$$|\overrightarrow{v}+\overrightarrow{w}| = 3\sqrt{19} \approx 13.07669683$$
$$|\overrightarrow{v}|+|\overrightarrow{w}| = 2\sqrt{79}+5\sqrt{5} \approx 28.95672871$$

(d) Is it true that $|\overrightarrow{v}\cdot\overrightarrow{w}| \leq |\overrightarrow{v}|\,|\overrightarrow{w}|$?

The answer is yes:

$$|\overrightarrow{v}\cdot\overrightarrow{w}| = \sqrt{19669} \approx 140.2462121$$
$$|\overrightarrow{v}|\,|\overrightarrow{w}| = 10\sqrt{395} \approx 198.7460691$$

10. Find a vector $\overrightarrow{w}$ of norm seven perpendicular to $\overrightarrow{v} = \langle -5, -9\rangle$.

First we find a vector $\overrightarrow{w}$ perpendicular to $\overrightarrow{v}$. Notice that $\overrightarrow{w} = \langle 9, -5\rangle$ is such a vector, but is by no means unique. To make this vector length seven, we divide it by its magnitude, then multiply by seven:

$$\overrightarrow{x} = \frac{7}{|\overrightarrow{w}|}\overrightarrow{w} = \frac{7}{\sqrt{106}}\langle 9, -5\rangle$$

11. Find two different vectors $\overrightarrow{w}$ of norm four perpendicular to $\overrightarrow{v} = \langle 7, 3, -6\rangle$.

Two simple vectors to pick for this problem would be

$$\overrightarrow{w_1} = \langle 0,6,3\rangle,\ \overrightarrow{w_2} = \langle -3,7,0\rangle$$

Clearly there are others, but these two are simple enough. We make them have norm four the same way as the previous problem:

$$\overrightarrow{x_1} = \frac{4}{3\sqrt{5}}\langle 0,6,3\rangle,\ \overrightarrow{x_2} = \frac{4}{\sqrt{58}}\langle -3,7,0\rangle$$

12. Find two nonzero vectors $\overrightarrow{w_1}$ and $\overrightarrow{w_2}$ perpendicular to $\overrightarrow{v} = \langle 7,3,-6\rangle$, where $\overrightarrow{w_1}$ and $\overrightarrow{w_2}$ are also perpendicular.

We take the first vector from the previous problem, $\overrightarrow{w_1} = \langle 0,6,3\rangle$, now we need to find a second vector, $\overrightarrow{w_2}$, such that $\overrightarrow{w_1}\cdot\overrightarrow{v} = \overrightarrow{w_2}\cdot\overrightarrow{v} = 0$. If we set $\overrightarrow{w_2} = \langle a,b,c\rangle$, then we have the two equations

$$\begin{aligned} 7a+3b-6c&=0\\ 6b-3c&=0 \end{aligned}$$

Solving two equations in three unknowns yields an infinite number of solutions (in this case). As an example, one solution is we can let $a = 9$, $b = 7$ and $c = 14$, yielding $\overrightarrow{w_2} = \langle 9,7,14\rangle$ .

13. Let $\overrightarrow{v} = \langle a,b,c\rangle \in \mathbb{R}^3$ be fixed. What equation must $\overrightarrow{w} = \langle x,y,z\rangle \in \mathbb{R}^3$ satisfy for $\overrightarrow{w}$ and $\overrightarrow{v}$ to be perpendicular, and what does this equation represent in $\mathbb{R}^3$?

The equation is

$$\langle a,b,c\rangle\cdot\langle x,y,z\rangle = 0,$$

or

$$ax+by+cz=0,$$

which is the equation of a plane in $\mathbb{R}^3$ through the origin.

14. Find the equation of the plane in $\mathbb{R}^3$ that is perpendicular to the vector $\overrightarrow{v} = \langle 7,3,-6\rangle$ and goes through the origin $\langle 0,0,0\rangle$.

Using the information from problem 13, the equation is

$$\langle 7,3,-6\rangle\cdot\langle x,y,z\rangle = 0$$

or $7x+3y-6z=0$.

15. Find the equation of the plane in $\mathbb{R}^3$ that is perpendicular to the vector $\overrightarrow{v} = \langle 7,3,-6\rangle$ and goes through the point $P(-2,5,8)$.

In this instance, we require that

$$\langle 7, 3, -6\rangle \cdot (\langle x, y, z\rangle - \langle -2, 5, 8\rangle) = 0,$$

which simplifies to $47+7x+3y-6z = 0$, or in standard form $7x+3y-6z = -47$.

16. Use the dot product to show that the diagonals of a square are perpendicular.

We will assume that one corner of the square is at the origin, as all angles are relative to the vectors themselves, and not where the square is located. A square can be formed using two perpendicular vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, and as a result, the diagonals are $\overrightarrow{a} + \overrightarrow{b}$ and $\overrightarrow{a} - \overrightarrow{b}$. If these two diagonals are perpendicular, then their dot product should be zero:

$$\begin{aligned}\left(\overrightarrow{a} + \overrightarrow{b}\right) \cdot \left(\overrightarrow{a} - \overrightarrow{b}\right) &= \overrightarrow{a} \cdot \overrightarrow{a} - \overrightarrow{a} \cdot \overrightarrow{b} + \overrightarrow{b} \cdot \overrightarrow{a} - \overrightarrow{b} \cdot \overrightarrow{b} \\ &= \overrightarrow{a} \cdot \overrightarrow{a} - \overrightarrow{b} \cdot \overrightarrow{b} \\ &= |\overrightarrow{a}|^2 - \left|\overrightarrow{b}\right|^2\end{aligned}$$

but since this is a square, the length vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ are zero, $\left(\overrightarrow{a} + \overrightarrow{b}\right) \cdot \left(\overrightarrow{a} - \overrightarrow{b}\right) = 0$.

17. Find a formula for the angle between the two diagonals of a parallelogram in terms of the lengths of any two of its adjacent sides.

The two vectors we need to find the angle between are $\overrightarrow{a} + \overrightarrow{b}$ and $\overrightarrow{a} - \overrightarrow{b}$. Using equation 6.7, we have

$$\begin{aligned}\cos(\theta) &= \frac{\left(\overrightarrow{a} + \overrightarrow{b}\right) \cdot \left(\overrightarrow{a} - \overrightarrow{b}\right)}{\left|\overrightarrow{a} + \overrightarrow{b}\right| \left|\overrightarrow{a} - \overrightarrow{b}\right|} \\ &= \frac{|\overrightarrow{a}|^2 - \left|\overrightarrow{b}\right|^2}{\left|\overrightarrow{a} + \overrightarrow{b}\right| \left|\overrightarrow{a} - \overrightarrow{b}\right|}\end{aligned}$$

18. Let $f(x)$ and $g(x)$ be any two real continuous functions for $x \in [a, b]$. Define the dot product of $f(x)$ and $g(x)$ by

$$f(x) \cdot g(x) = \int_a^b f(x)\, g(x) dx$$

(a) Find $f(x) \cdot g(x)$, $f(x) \cdot f(x)$, and $g(x) \cdot g(x)$ for $f(x) = e^x$ and $g(x) = \cos(x)$ on the interval $[0, \pi]$.

$$\int_0^{\pi} f(x)\, g(x)\, dx = -\frac{1}{2}\left(e^{\pi} + 1\right)$$
$$\int_0^{\pi} f(x)\, f(x)\, dx = \frac{1}{2}\left(e^{2\,\pi} - 1\right)$$
$$\int_0^{\pi} g(x)\, g(x)\, dx = \frac{\pi}{2}$$

(b) Show that this dot product of two functions satisfies the usual properties of a real dot product.

Using the rules of integral calculus, we have

$$\int_a^b f(x)\, g(x)\, dx = \int_a^b g(x)\, f(x)\, dx$$
$$\int_a^b (f(x) + g(x))\, h(x)\, dx = \int_a^b f(x)\, h(x)\, dx + \int_a^b g(x)\, h(x)\, dx$$
$$\int_a^b f(x)\, f(x)\, dx = \int_a^b |f(x)|^2\, dx$$
$$\int_a^b (c\, f(x))\, g(x)\, dx = c \int_a^b f(x)\, g(x)\, dx$$
$$\int_a^b f(x)\, (d\, g(x))\, dx = d \int_a^b f(x)\, g(x)\, dx$$

(c) Find all possible dot products of the functions

$$\{1, \cos(x), \cos(2x), \sin(x), \sin(2x)\}$$

with each other and themselves on the interval $[0, 2\pi]$. Do you see a pattern here, what is it?

$$\int_0^{2\pi} 1\, dx = 2\pi$$
$$\int_0^{2\pi} \cos(x)\, \cos(x)\, dx = \int_0^{2\pi} \cos(2x)\, \cos(2x)\, dx = \pi$$
$$\int_0^{2\pi} \sin(x)\, \sin(x)\, dx = \int_0^{2\pi} \sin(2x)\, \sin(2x)\, dx = \pi$$
$$\int_0^{2\pi} 1\, \cos(x)\, dx = \int_0^{2\pi} 1\, \cos(2x)\, dx = 0$$

$$\int_0^{2\pi} 1\,\sin(x)\,dx = \int_0^{2\pi} 1\,\sin(2x)\,dx = 0$$
$$\int_0^{2\pi} \cos(x)\,\cos(2x)\,dx = \int_0^{2\pi} \cos(x)\,\sin(x)\,dx = 0$$
$$\int_0^{2\pi} \cos(x)\,\sin(2x)\,dx = \int_0^{2\pi} \cos(2x)\,\sin(x)\,dx = 0$$
$$\int_0^{2\pi} \cos(2x)\,\sin(2x)\,dx = \int_0^{2\pi} \sin(x)\,\sin(2x)\,dx = 0$$

Since the dot product between any two distinct functions in this set is zero, we have a set of perpendicular functions.

(d) Find all possible dot products of the functions

$$\{1, x, x^2, x^3\}$$

with each other and themselves on the interval $[-1, 1]$.

$$\int_{-1}^{1} 1\cdot 1\,dx = 2, \quad \int_{-1}^{1} 1\cdot x^2\,dx = \frac{2}{3}, \quad \int_{-1}^{1} 1\cdot x\,dx = \int_{-1}^{1} 1\cdot x^3\,dx = 0$$
$$\int_{-1}^{1} x\,x\,dx = \frac{2}{3}, \quad \int_{-1}^{1} x\,x^2\,dx = 0, \quad \int_{-1}^{1} x\,x^3\,dx = \frac{2}{5}$$
$$\int_{-1}^{1} x^2\,x^2\,dx = \frac{2}{5}, \quad \int_{-1}^{1} x^2\,x^3\,dx = 0, \quad \int_{-1}^{1} x^3\,x^3\,dx = \frac{2}{7}$$

19. Let $A$ be a real $n \times n$ matrix with $\overrightarrow{u}, \overrightarrow{v} \in \mathbb{R}^n$ written as column vectors in $\mathbb{R}^n$. Show that the real dot product satisfies $(A\overrightarrow{u}) \cdot \overrightarrow{v} = \overrightarrow{u} \cdot \left(A^T\,\overrightarrow{v}\right)$. Give an example when $n = 2$ to illustrate this formula.

By the definition of the dot product, we have

$$\begin{aligned}(A\overrightarrow{u}) \cdot \overrightarrow{v} &= (A\overrightarrow{u})^T\,\overrightarrow{v} \\ &= \left(\overrightarrow{u}^T\,A^T\right)\,\overrightarrow{v} \\ &= \overrightarrow{u}^T\,\left(A^T\,\overrightarrow{v}\right) \\ &= \overrightarrow{u} \cdot \left(A^T\,\overrightarrow{v}\right)\end{aligned}$$

As an example, we will will let

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}, \; \overrightarrow{u} = \langle 3, 3\rangle, \; \overrightarrow{v} = \langle -1, 5\rangle$$

and thus

$$\begin{aligned}(A\vec{u})\cdot\vec{v} &= \langle 9,-6\rangle\cdot\langle -1,5\rangle = -39\\ \vec{u}\cdot\left(A^T\,\vec{v}\right) &= \langle 3,3\rangle\cdot\langle -16,3\rangle = -39\end{aligned}$$

20. Let $A$ be a complex $n\times n$ matrix with $\vec{u},\vec{v}\in\mathbb{C}^n$ written as column vectors in $\mathbb{C}^n$. Show that the complex dot product satisfies $(A\vec{u})\cdot\vec{v} = \vec{u}\cdot\left(\overline{A}^T\,\vec{v}\right)$. Give an example when $n=2$ to illustrate this formula.

By the definition of the dot product, we have

$$\begin{aligned}(A\vec{u})\cdot\vec{v} &= (A\vec{u})^T\,\overline{\vec{v}}\\ &= \left(\vec{u}^T A^T\right)\overline{\vec{v}}\\ &= \vec{u}^T\left(A^T\,\overline{\vec{v}}\right)\\ &= \vec{u}^T\left(\overline{\overline{A^T}\,\vec{v}}\right)\\ &= \vec{u}^T\left(\overline{\overline{A}^T\,\vec{v}}\right)\\ &= \vec{u}\cdot\left(\overline{A}^T\,\vec{v}\right)\end{aligned}$$

As an example, we will will let

$$A = \begin{bmatrix} 1+i & 2-3\,i\\ -3+2\,i & 4+6\,i\end{bmatrix},\ \vec{u} = \langle 2+2\,i, 3-2\,i\rangle,\ \vec{v} = \langle -1+2\,i, 5-3\,i\rangle$$

and thus

$$\begin{aligned}(A\vec{u})\cdot\vec{v} &= \langle -9\,i, 14+8\,i\rangle\cdot\langle 1+2\,i, 5-3\,i\rangle = 28+91\,i\\ \vec{u}\cdot\left(\overline{A}^T\,\vec{v}\right) &= \langle 2+2\,i, 3-2\,i\rangle\cdot\langle -20-2\,i, -6+41\,i\rangle = 28+91\,i\end{aligned}$$

## 6.3 Cross Product

1. Given $\vec{u} = \langle 1,2,-1\rangle$, $\vec{v} = \langle -3,-1,4\rangle$, and $\vec{w} = \langle 5,-1,0\rangle$, compute the following. As previously defined, $\vec{i} = \langle 1,0,0\rangle$, $\vec{j} = \langle 0,1,0\rangle$, and $\vec{k} = \langle 0,0,1\rangle$ are the three standard unit vectors of $\mathbb{R}^3$.

(a) $\overrightarrow{v} \times \overrightarrow{w} = \langle 4, 20, 8\rangle$ (b) $\overrightarrow{w} \times \overrightarrow{u} = \langle 1, 5, 11\rangle$

(c) $\overrightarrow{u} \times \overrightarrow{v} = \langle 7, -1, 5\rangle$ (d) $\overrightarrow{w} \times \overrightarrow{v} = \langle -4, -20, -8\rangle$

(e) $(\overrightarrow{u} \times \overrightarrow{v}) \times \overrightarrow{w} = \langle 5, 25, -2\rangle$ (f) $\overrightarrow{u} \times (\overrightarrow{v} \times \overrightarrow{w}) = \langle 36, -12, 12\rangle$

(g) $\overrightarrow{i} \times \overrightarrow{v} = \langle 0, -4, -1\rangle$ (h) $\overrightarrow{u} \times \overrightarrow{k} = \langle 2, -1, 0\rangle$

(i) $\overrightarrow{j} \times \overrightarrow{w} = \langle 0, 0, -5\rangle$

2. Compute the areas of each of the triangles defined by the following sets of points.

(a) $\{(0, 1), (2, 3), (6, 2)\}$

First we need two displacement vectors, $\overrightarrow{v}$ and $\overrightarrow{w}$. We can choose any point as the common base point, so we pick $(0, 1)$. This gives $\overrightarrow{v} = \langle 2, 2\rangle$ and $\overrightarrow{w} = \langle 6, 1\rangle$. Using equation 6.13, we have

$$\begin{aligned} A &= \frac{1}{2} \left|\langle 2, 2, 0\rangle \times \langle 6, 1, 0\rangle\right| \\ &= \frac{1}{2} \left|\langle 0, 0, -10\rangle\right| \\ &= 5 \end{aligned}$$

(b) $\{(0, 1, 2), (2, 3, 1), (1, 6, 2)\}$

Similar to part (a), we use the first point as base point, to get $\overrightarrow{v} = \langle 2, 2, -1\rangle$ and $\overrightarrow{w} = \langle 1, 5, 0\rangle$.

$$\begin{aligned} A &= \frac{1}{2} \left|\langle 2, 2, -1\rangle \times \langle 1, 5, 0\rangle\right| \\ &= \frac{1}{2} \left|\langle 5, 1, -8\rangle\right| \\ &= \frac{1}{2} 3\sqrt{10} \end{aligned}$$

3. Prove the following properties of the cross product.

$$\text{(a)}\ \overrightarrow{i} \times \overrightarrow{j} = \det\left(\begin{bmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right) = 0\,\overrightarrow{i} + 0\,\overrightarrow{j} + 1\,\overrightarrow{k} = \overrightarrow{k}$$

$$\text{(b)}\ \overrightarrow{j} \times \overrightarrow{k} = \det\left(\begin{bmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 1\,\overrightarrow{i} + 0\,\overrightarrow{j} + 0\,\overrightarrow{k} = \overrightarrow{i}$$

$$(c)\, \overrightarrow{k} \times \overrightarrow{i} = \det\left(\begin{bmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\right) = 0\,\overrightarrow{i} + 1\,\overrightarrow{j} + 0\,\overrightarrow{k} = \overrightarrow{j}$$

$$(d)\, \overrightarrow{j} \times \overrightarrow{i} = \det\left(\begin{bmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right) = 0\,\overrightarrow{i} + 0\,\overrightarrow{j} - 1\,\overrightarrow{k} = -\overrightarrow{k}$$

$$(e)\, \overrightarrow{k} \times \overrightarrow{j} = \det\left(\begin{bmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\right) = -1\,\overrightarrow{i} + 0\,\overrightarrow{j} + 0\,\overrightarrow{k} = -\overrightarrow{i}$$

$$(f)\, \overrightarrow{i} \times \overrightarrow{k} = \det\left(\begin{bmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0\,\overrightarrow{i} - 1\,\overrightarrow{j} + 0\,\overrightarrow{k} = -\overrightarrow{j}$$

4. If $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w} \in \mathbb{R}^3$, and $\alpha \in \mathbb{R}$, verify the following identities: (Property (c) is an example of a *Jacobi identity*).

(a) $\overrightarrow{u} \times (\overrightarrow{v} + \overrightarrow{w}) = (\overrightarrow{u} \times \overrightarrow{v}) + (\overrightarrow{u} \times \overrightarrow{w})$

This is a straight forward computation.

(b) $(\alpha\overrightarrow{u}) \times \overrightarrow{v} = \overrightarrow{u} \times (\alpha\overrightarrow{v})$

This is a straight forward computation.

(c) $\overrightarrow{u} \times (\overrightarrow{v} \times \overrightarrow{w}) + \overrightarrow{v} \times (\overrightarrow{w} \times \overrightarrow{u}) + \overrightarrow{w} \times (\overrightarrow{u} \times \overrightarrow{v}) = 0$

This is a straight forward computation, albeit a little lengthy.

5. If $\overrightarrow{v(t)} = \langle \cos(t), \sin(t), 0\rangle$, show that the angle $\theta$ between $\overrightarrow{v(t)}$ and $\overrightarrow{i}$ satisfies $\sin(\theta) = |\sin(t)|$, and that $\left|\overrightarrow{v(t)} \times \overrightarrow{i}\right|^2 = \sin^2(t)$.

Using equation 6.12, we have

$$\begin{aligned}\sin(\theta) &= \frac{\left|\overrightarrow{v} \times \overrightarrow{i}\right|}{|\overrightarrow{v}|\;|\overrightarrow{v}|} \\ &= \frac{|\langle 0, 0, -\sin(t)\rangle|}{1 \cdot 1} \\ &= |\sin(t)|\end{aligned}$$

However since $0 \leq \theta \leq \pi$, $\sin(\theta) \geq 0$, we have $\sin(\theta) = |\sin(t)|$. Furthermore, we automatically arrive at $\left|\overrightarrow{v(t)} \times \overrightarrow{i}\right|^2 = \sin^2(t)$.

6. Verify the two cross product and dot product properties in the table at the end of the section.

This is another straight forward computation, similar to problem 4.

7. (a) Let $ax + by + cz = d$ be the equation of a plane. Show that for any two points $P$ and $Q$ in this plane, the displacement vector $\overrightarrow{PQ}$ is perpendicular to $\overrightarrow{v} = \langle a, b, c\rangle$, that is, $\overrightarrow{v} = \langle a, b, c\rangle$ is perpendicular to the plane $ax+by+cz = d$.

If we set $P = (x_p, y_p, z_p)$ and $Q = (x_q, y_q, z_q)$, then

$$a\,x_p + b\,y_p + c\,z_p = d,\;\; a\,x_q + b\,y_q + c\,z_q = d$$

and taking the difference of these two equations yields

$$a\;(x_q - x_p) + b\;(y_q - y_p) + c\;(z_q - z_p) = d - d$$

or

$$\langle a, b, c\rangle \cdot \overrightarrow{PQ} = 0$$

(b) Let $ax + by + cz = d$ and $ex + fy + gz = h$ be two intersecting planes. Find the equation of the plane through the origin perpendicular to these two given planes.

Setting $\overrightarrow{p} = \langle a, b, c\rangle$ and $\overrightarrow{q} = \langle e, f, g\rangle$, notice that these vectors are perpendicular points lying on each plane, respectively. Therefore, the equation of the plane we desire is given by

$$(\overrightarrow{p} \times \overrightarrow{q}) \cdot \langle x, y, z\rangle = 0$$

(c) Let $5x + 2y + 7z = -9$ and $3x - 4y + 11z = 1$ be two intersecting planes. Find the equation of the plane through the origin perpendicular to these two given planes. Find the equation of the plane through $P(-1, 0, 4)$ perpendicular to these two given planes.

Using the formula from the previous problem, we have

$$\begin{aligned}(\langle 5,2,7\rangle \times \langle 3,-4,11\rangle)\cdot\langle x,y,z\rangle &= 0\\ \langle 50,-34,-26\rangle\cdot\langle x,y,z,\rangle &= 0\\ 50\,x-34\,y-26\,z &= 0\end{aligned}$$

and the plane passing through the point $P(-1,0,4)$ is given by

$$\begin{aligned}(\langle 5,2,7\rangle \times \langle 3,-4,11\rangle)\cdot\langle x+1,y,z-4\rangle &= 0\\ 50\,x-34\,y-26\,z &= 154\end{aligned}$$

8. (a) Let $x = a\,t+\alpha$, $y = b\,t+\beta$, $z = c\,t+\gamma$ be the parametric equation of a line in space for $t \in \mathbb{R}$. Show that for any two points $P$ and $Q$ on this line that the displacement vector $\overrightarrow{PQ}$ is parallel to $\overrightarrow{v} = \langle a,b,c\rangle$, that is, $\overrightarrow{v} = \langle a,b,c\rangle$ is parallel to the line $x = a\,t+\alpha$, $y = b\,t+\beta$, $z = c\,t+\gamma$. Note that when $t = 0$, we have that $(\alpha,\beta,\gamma)$ is a point on this line. As well, this parametric equation for a line in space is the spacial version of the point-slope formula for a line in the $xy$-plane.

Let $t_0$ be the time corresponding to point $P$, and $t_1$ to $Q$. Thus,

$$P = (a\,t_0+\alpha, b\,t_0+\beta, c\,t_0+\gamma)\,,\; Q = (a\,t_1+\alpha, b\,t_1+\beta, c\,t_1+\gamma)$$

and

$$\begin{aligned}\overrightarrow{PQ} &= \langle a\ (t_1-t_0)\,, b\ (t_1-t_0)\,, c\ (t_1-t_0)\rangle\\ &= (t_1-t_0)\ \langle a,b,c\rangle\\ &= (t_1-t_0)\ \overrightarrow{v}\end{aligned}$$

(b) Let $px+qy+rz = s$ and $ex+fy+gz = h$ be two intersecting planes. Find a vector parallel to their line of intersection.

A vector parallel to to the line of intersection between planes is simply

$$\langle p,q,r\rangle \times \langle e,f,g\rangle$$

since the vectors in the above cross product are perpendicular to their planes, the cross product must be perpendicular to those perpendicular vectors, and must therefore be parallel to the line of intersection.

(c) Let $5x+2y+7z = -9$ and $3x-4y+11z = 1$ be two intersecting planes. Find the parametric equation for their line of intersection.

Using part (b), the vector parallel to the line of intersection is given by $\langle 5,2,7\rangle \times \langle 3,-4,11\rangle = \langle 50,-34,-26\rangle$. Now we need to find a point that both planes share in common. Since we are attempting to solve two equations with three unknowns, we expect a solution of dimension one (i.e. a line). So to find only a point, we select a value for one of the variables. We will set $z = 0$ and see if we get a solution:

$$\begin{aligned} 5x + 2y &= -9 \\ 3x - 4y &= 1 \end{aligned}$$

has solution $x = -\frac{17}{13}, y = -\frac{16}{13}$. Therefore, the point $\left(-\frac{17}{13}, -\frac{16}{13}, 0\right)$ lies on the line of intersection. We now have a point and a direction. Thus, if we start at the point just found, and travel in the direction of the cross product (any distance at all), we are still on the line of intersection. Putting this into a mathematical expression, we get that the line of intersection can be parameterized as

$$\langle x,y,z\rangle = \left\langle -\frac{17}{13} + 50\,t, -\frac{16}{13} - 34\,t, 0 - 26\,t \right\rangle.$$

Once again note that there are many ways to parameterize this line, as you can chose any point on the line and any vector in the direction parallel to the line.

9. Using problem 7, find the equation of the plane through the three points $P(1,5,9)$, $Q(-3,4,-8)$, and $R(7,-2,6)$.

First we create two displacement vectors $\overrightarrow{v} = Q - P = \langle -4,-1,-17\rangle$ and $\overrightarrow{w} = R - P = \langle 6,-7,-3\rangle$ and then we take the cross product of the two:

$$\langle -4,-1,-17\rangle \times \langle 6,-7,-3\rangle = \langle -116,-114,34\rangle$$

which is perpendicular to the plane in question. Using the same ideas as part (c) of problem 7, we get the formula for the plane is given by

$$\langle -116,-114,34\rangle \cdot \langle x-1, y-5, z-9\rangle = 0$$

where we have chosen the point $P$ as the point on the plane (we could have chosen $Q$ or $R$). Expanding this equation gives the standard form of this plane to be $-116\,x - 114\,y + 34\,z = -380$.

10. (See problem 8.) (a) Let $x = a\,t+\alpha$, $y = b\,t+\beta$, $z = c\,t+\gamma$ and $x = d\,t+\delta$, $y = e\,t+\theta$, $z = f\,t+\lambda$ be two intersecting lines in space. Find a vector perpendicular to the plane through these two points.

The direction of the first line is $\langle a,b,c\rangle$ and the second is $\langle d,e,f\rangle$. Therefore, a vector perpendicular to both is the cross product of the two just given:

$$\langle a,b,c\rangle \times \langle d,e,f\rangle$$

(b) Find the equation of the plane through the two intersecting lines $x = 5t + 2, y = -3t + 1, z = 4t - 5$ and $x = -7t + 2, y = 2t + 1, z = 9t - 5$.

First, we need the vector perpendicular to both lines. Using part (a), we see that this vector is

$$\langle 5, -3, 4\rangle \times \langle -7, 2, 9\rangle = \langle -35, -73, -11\rangle$$

Now we need to find the point of intersection of these two lines. Setting each pair of coordinates equal yields the system of equations

$$\{5\,t + 2 = -7\,t + 2,\ -3\,t + 1 = 2\,t + 1,\ 4\,t - 5 = 9\,t - 5\}$$

which has solution $t = 0$. Plugging in $t = 0$ into either line gives the point of intersection to be $(2, 1, -5)$. Now we have a point on the plane, and the perpendicular vector, so our final answer is

$$\langle -35, -73, -11\rangle \cdot \langle x - 2, y - 1, z + 5\rangle = 0$$

or in standard form $-35\,x - 73\,y - 11\,z = -88$.

11. Explain why the associative property of the cross product is false, that is, explain why

$$\overrightarrow{u} \times (\overrightarrow{v} \times \overrightarrow{w}) \neq (\overrightarrow{u} \times \overrightarrow{v}) \times \overrightarrow{w}$$

Give a general example to illustrate that this is correct.

The argument is simply a geometric one. Try to draw these vectors in your head. Note that $\overrightarrow{v} \times \overrightarrow{w}$ is perpendicular to both $\overrightarrow{v}$ and $\overrightarrow{w}$ Crossing this with $\overrightarrow{u}$ gives a vector perpendicular to both $\overrightarrow{v} \times \overrightarrow{w}$ and $\overrightarrow{u}$.

On the other hand, the second set of cross products will result in a vector which is perpendicular to $\overrightarrow{u} \times \overrightarrow{v}$ and $\overrightarrow{w}$ instead.

As an example consider $\overrightarrow{u} = \langle 1, 0, -1\rangle$, $\overrightarrow{v} = \langle 0, 1, 0\rangle$ and $\overrightarrow{w} = \langle 1, 1, 1\rangle$, then

$$\overrightarrow{u} \times (\overrightarrow{v} \times \overrightarrow{w}) = \langle 0, 0, 0\rangle$$
$$(\overrightarrow{u} \times \overrightarrow{v}) \times \overrightarrow{w} = \langle -1, 0, 1\rangle$$

Note that the first answer was the zero vector. This example was constructed to show that in the first formula, the first cross product yields a vector in the direction of the third, which results in the second cross product in the expression yielding the zero vector. In the second formula, the first cross product does not result in a vector in the direction of the third, which implies the second cross product in the expression will result in a nonzero vector.

12. Can the cross product be defined for complex vectors in $\mathbb{C}^3$? If no, explain why not. If yes, then do all of the properties of the cross product still hold if we switch to complex dot product?

Unlike the dot product where conjugates need to be taken, nothing changes with vectors in $\mathbb{C}^3$ and yes, all formulas still hold.

## 6.4 Vector Projection

1. For $\overrightarrow{v}, \overrightarrow{w} \in \mathbb{R}^n$, under what conditions does $\text{proj}_{\overrightarrow{v}}(\overrightarrow{w}) = \overrightarrow{0}$?

For the projection to be zero, the two vectors must be perpendicular (draw the picture in your head, see why this is so).

2. For $\overrightarrow{v}, \overrightarrow{w} \in \mathbb{R}^n$, relate the sign of $\text{comp}_{\overrightarrow{v}}(\overrightarrow{w})$ to the angle $\theta$ between $\overrightarrow{v}$ and $\overrightarrow{w}$.

If the angle satisfies $0 \leq \theta \leq \frac{\pi}{2}$, then $\text{comp}_{\overrightarrow{v}}(\overrightarrow{w}) \geq 0$ (see figure 6.12). If $\frac{\pi}{2} < \theta \leq \pi$, then $\text{comp}_{\overrightarrow{v}}(\overrightarrow{w}) \leq 0$ (see figure 6.13).

3. Compute $\text{comp}_{\overrightarrow{v}}(\overrightarrow{w})$ for the following pairs of vectors:

Refer to equation 6.19 for the formula:

(a) $\overrightarrow{v} = \langle -1, 2\rangle, \overrightarrow{w} = \langle 3, 5\rangle \to \text{comp}_{\overrightarrow{v}}(\overrightarrow{w}) = \frac{7}{\sqrt{5}}$

(b) $\overrightarrow{v} = \langle 4, 6\rangle, \overrightarrow{w} = \langle 2, 3\rangle \to \text{comp}_{\overrightarrow{v}}(\overrightarrow{w}) = \sqrt{13}$

(c) $\overrightarrow{v} = \langle 3, 0\rangle, \overrightarrow{w} = \langle 5, 1\rangle \to \text{comp}_{\overrightarrow{v}}(\overrightarrow{w}) = 5$

(d) $\overrightarrow{v} = \langle -1, 0, 2\rangle, \overrightarrow{w} = \langle 3, 2, -2\rangle \to \text{comp}_{\overrightarrow{v}}(\overrightarrow{w}) = -\frac{7}{\sqrt{5}}$

(e) $\overrightarrow{v} = \langle 1, 1, -1\rangle, \overrightarrow{w} = \langle 2, -1, 2\rangle \to \text{comp}_{\overrightarrow{v}}(\overrightarrow{w}) = -\frac{1}{\sqrt{3}}$

(f) $\overrightarrow{v} = \langle 3, 0, 0\rangle, \overrightarrow{w} = \langle -1, 1, 1\rangle \to \text{comp}_{\overrightarrow{v}}(\overrightarrow{w}) = -1$

(g) $\overrightarrow{v} = \langle 3, -2, -4\rangle, \overrightarrow{w} = \langle -1, 2, 0\rangle \to \text{comp}_{\overrightarrow{v}}(\overrightarrow{w}) = -\frac{7}{\sqrt{29}}$

(h) $\overrightarrow{v} = \langle 1, 1, -1, 1\rangle, \overrightarrow{w} = \langle 1, -1, 1, 1\rangle \to \text{comp}_{\overrightarrow{v}}(\overrightarrow{w}) = 0$

4. Compute $\text{proj}_{\overrightarrow{v}}(\overrightarrow{w})$ for the following pairs of vectors:

Notice that these are the same vectors as those from problem 3. We will use the formula

$$\text{proj}_{\overrightarrow{v}}(\overrightarrow{w}) = \text{comp}_{\overrightarrow{v}}(\overrightarrow{w})\frac{\overrightarrow{v}}{|\overrightarrow{v}|}$$

which means all we have to do is find the unit length vector for each $\overrightarrow{v}$ in the given problems, and multiply them by the corresponding answers to problem 3.

(a) $\overrightarrow{v} = \langle -1, 2\rangle, \overrightarrow{w} = \langle 3, 5\rangle \to \text{proj}_{\overrightarrow{v}}(\overrightarrow{w}) = \left\langle -\frac{7}{5}, \frac{14}{5}\right\rangle$

(b) $\overrightarrow{v} = \langle 4, 6\rangle, \overrightarrow{w} = \langle 2, 3\rangle \to \text{proj}_{\overrightarrow{v}}(\overrightarrow{w}) = \langle 2, 3\rangle$

(c) $\overrightarrow{v} = \langle 3, 0\rangle, \overrightarrow{w} = \langle 5, 1\rangle \to \text{proj}_{\overrightarrow{v}}(\overrightarrow{w}) = \langle 5, 0\rangle$

(d) $\overrightarrow{v} = \langle -1, 0, 2\rangle, \overrightarrow{w} = \langle 3, 2, -2\rangle \to \text{proj}_{\overrightarrow{v}}(\overrightarrow{w}) = \left\langle \frac{7}{5}, 0, -\frac{14}{5}\right\rangle$

(e) $\overrightarrow{v} = \langle 1, 1, -1\rangle, \overrightarrow{w} = \langle 2, -1, 2\rangle \to \text{proj}_{\overrightarrow{v}}(\overrightarrow{w}) = \left\langle -\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right\rangle$

(f) $\overrightarrow{v} = \langle 3, 0, 0\rangle, \overrightarrow{w} = \langle -1, 1, 1\rangle \to \text{proj}_{\overrightarrow{v}}(\overrightarrow{w}) = \langle -1, 0, 0\rangle$

(g) $\overrightarrow{v} = \langle 3, -2, -4\rangle, \overrightarrow{w} = \langle -1, 2, 0\rangle \to \text{proj}_{\overrightarrow{v}}(\overrightarrow{w}) = \left\langle -\frac{21}{29}, \frac{14}{29}, \frac{28}{29}\right\rangle$

(h) $\overrightarrow{v} = \langle 1, 1, -1, 1\rangle, \overrightarrow{w} = \langle 1, -1, 1, 1\rangle \to \text{proj}_{\overrightarrow{v}}(\overrightarrow{w}) = \langle 0, 0, 0, 0\rangle$

5. Compute the normal vectors to each of the following planes:

(a) $3x - 5y = 7 \to \overrightarrow{n} = \langle 3, -5\rangle$

(b) $2x + 8y = 2 \to \overrightarrow{n} = \langle 2, 8\rangle$

(c) $x - 5y + 7z = 3 \to \overrightarrow{n} = \langle 1, -5, 7\rangle$

(d) $2w - 4x + 5y - 7z = 2 \to \overrightarrow{n} = \langle 2, -4, 5, -7\rangle$

6. Find the distance from the point $P(2, 3)$ to the line $3x - 4y = 6$.

Using equation 6.20, we have that

$$D(P, L) = \frac{|(3)(2) + (-4)(3) - 6|}{\sqrt{3^2 + (-4)^2}} = \frac{12}{5}$$

7. Find the distance from the point $P(1, -2, 4)$ to the plane $2x + 5y - 6z = 1$.

Using equation 6.22, we have that

$$D(P, R) = \frac{|(2)(1) + (5)(-2) + (-6)(4) - 1|}{\sqrt{2^2 + 5^2 + (-6)^2}} = \frac{33}{\sqrt{65}}$$

8. Find the distance from the point $P(1,-2,4,-1)$ to the plane $w+3x-2y+2z=-1$.

Modifying equation 6.20 or 6.22, we have

$$\begin{aligned} D(P,R) &= \frac{|a\,w_0+b\,x_0+c\,y_0+d\,z_0-e|}{\sqrt{a^2+b^2+c^2+d^2}} \\ &= \frac{|(1)(1)+(3)(-2)+(-2)(4)+(2)(-1)-(-1)|}{\sqrt{1^2+3^2+(-2)^2+2^2}} \\ &= \frac{14}{3\sqrt{2}} \end{aligned}$$

9. For each of problems 6-8, find the point that lies on the line or plane at the location that minimizes distance between the line or plane and $P$.

(a) We do this two ways. First note that since we have the distance from the point to the line being $\frac{12}{5}$, we simply need to start at the point $P$ and go $\frac{12}{5}$ units in the direction perpendicular to the line starting at $P$. This will yield the point on the line, as desired. The direction perpendicular to $3x-4y=6$ is $\langle 3,-4\rangle$. This our closest point $Q$, on the line, can be gotten by the formula:

$$\overrightarrow{Q} = \langle 2,3\rangle + \frac{12}{5}\frac{\langle 3,-4\rangle}{|\langle 3,-4\rangle|} = \left\langle \frac{86}{25}, \frac{27}{25}\right\rangle$$

Secondly, we could find the intersection of the line $3x-4y=6$, and the line perpendicular to it which passes through $P(2,3)$. The equation of this perpendicular line is $4\,(x-2)+3\,(y-3)=0$. Solving for $x$ and $y$ gives $x=\frac{86}{25}$, and $y=\frac{27}{25}$, which agrees with the first approach.

(b) We will use the first approach from part (a) here:

$$\overrightarrow{Q} = \langle 1,-2,4\rangle + \frac{33}{\sqrt{65}}\frac{\langle 2,5,-6\rangle}{|\langle 2,5,-6\rangle|} = \left\langle \frac{131}{65}, \frac{7}{13}, \frac{62}{65}\right\rangle$$

(c) Same here

$$\overrightarrow{Q} = \langle 1,-2,4,-1\rangle + \frac{14}{3\sqrt{2}}\frac{\langle 1,3,-2,2\rangle}{|\langle 1,3,-2,2\rangle|} = \left\langle \frac{16}{9}, \frac{1}{3}, \frac{22}{9}, \frac{5}{9}\right\rangle$$

10. Can the ideas of this section be generalized to vector projection of complex vectors in $\mathbb{C}^n$? If yes, then explain how and what still works using the complex dot product. If no, then explain why not specifically stating what fails.

From the text, we know that $\text{proj}_{\overrightarrow{v}}(\overrightarrow{w})$ is simply a vector in the direction of $\overrightarrow{v}$, thus $\text{proj}_{\overrightarrow{v}}(\overrightarrow{w}) = \alpha\,\overrightarrow{v}$, for a complex scalar $\alpha$. The requirement is simply that $\overrightarrow{v} \cdot (\overrightarrow{w} - \alpha\,\overrightarrow{v}) = 0$. Working with this equation gives:

$$\begin{aligned}
\overrightarrow{v} \cdot (\overrightarrow{w} - \alpha\,\overrightarrow{v}) &= 0 \\
\overrightarrow{v} \cdot \overrightarrow{w} &= \overrightarrow{v} \cdot (\alpha\,\overrightarrow{v}) \\
\overrightarrow{v} \cdot \overrightarrow{w} &= \overline{\alpha}\,(\overrightarrow{v} \cdot \overrightarrow{v}) \\
\frac{\overrightarrow{v} \cdot \overrightarrow{w}}{|\overrightarrow{v}|^2} &= \overline{\alpha} \\
\frac{\overrightarrow{w} \cdot \overrightarrow{v}}{|\overrightarrow{v}|^2} &= \alpha
\end{aligned}$$

Some of these steps required specific properties of the complex dot product, so please be sure you follow all steps given above.

As an example, consider $\overrightarrow{v} = \langle 1+i, 1-2\,i\rangle$ and $\overrightarrow{w} = \langle 2+3\,i, -4+i\rangle$. Using the definition of $\alpha$ given above, we get that $\alpha = -\frac{1}{7} - \frac{6}{7}\,i$. Thus, $\text{proj}_{\overrightarrow{v}}(\overrightarrow{w}) = \langle \frac{5}{7} - i, -\frac{13}{7} - \frac{4}{7}\,i\rangle$. Clearly $\text{proj}_{\overrightarrow{v}}(\overrightarrow{w})$ is in the direction of $\overrightarrow{v}$ since $\text{proj}_{\overrightarrow{v}}(\overrightarrow{w}) = \left(-\frac{1}{7} - \frac{6}{7}\,i\right)\overrightarrow{v}$. We also get that $\overrightarrow{v} \cdot (\overrightarrow{w} - \text{proj}_{\overrightarrow{v}}(\overrightarrow{w})) = 0$. These are the two requirements of the projection vector, and therefore we have successfully performed a complex vector projection.

11. Find a formula for the shortest distance between the two parallel planes $ax + by + cz = d$ and $ax + by + cz = e$. Test your formula on an example of two planes.

To find a formula for the distance between these planes, all we need to do is chose a point on one plane, and then apply the standard formula (equation 6.22) for the distance from a point to a plane.

To find a point on the plane $ax + by + cz = d$, we will assume that $c \neq 0$, and let $x = y = 0$, in which case $z = \frac{d}{c}$. So the point is now given to be $\left(0, 0, \frac{d}{c}\right)$.

Using equation 6.22, with $P\left(0, 0, \frac{d}{c}\right)$ and $ax + by + cz = e$ gives

$$\begin{aligned}
D(P, R) &= \frac{\left|(a)(0) + (b)(0) + (c)\left(\frac{d}{c}\right) - e\right|}{\sqrt{a^2 + b^2 + c^2}} \\
&= \frac{|d - e|}{\sqrt{a^2 + b^2 + c^2}}
\end{aligned}$$

As an example of two planes, consider $2\,x - 3\,y + 4\,z = 1$ and $4\,x - 6\,y + 8\,z = -4$. Note that the second plane can be written as $2\,x - 3\,y + 4\,z = -2$. If we let$x = y = 0$ in the first equation, then solving for $z$ gives $z = \frac{1}{4}$. So the point

$P$ is given as $P\left(0,0,\frac{1}{4}\right)$. We now plug this into the formula to get

$$\begin{aligned} D(P,R) &= \frac{\left|(2)(0)+(-3)(0)+(4)\left(\frac{1}{4}\right)-(-2)\right|}{\sqrt{2^2+(-3)^2+4^2}} \\ &= \frac{3}{\sqrt{29}} \end{aligned}$$

12. Find a formula for the shortest distance between the line $x = at + \alpha$ , $y = bt+\beta$, $z = ct+\gamma$, parallel to the plane $ex+fy+gz = h$. Test your formula on an example of a parallel line and plane. Also, how can we test if a line and plane in $\mathbb{R}^3$ are parallel or not?

Since any point will do, we set $t = 0$ in the parametric equation of the line to get $P(\alpha, \beta, \gamma)$. Then we apply equation 6.22 to get

$$D(P,R) = \frac{|(e)(\alpha)+(f)(\beta)+(g)(\gamma)-h|}{\sqrt{e^2+f^2+g^2}}$$

To determine if the line and plane are parallel (and hence do not intersect), we first recognize that the vector perpendicular to the plane is $\overrightarrow{n} = \langle e, f, g\rangle$. The direction of the line is given by $\langle a, b, c\rangle$ (why?), and all we now have to do is take the dot product of the two and make sure the result is zero:

$$\langle e, f, g\rangle \cdot \langle a, b, c\rangle = 0$$

13. Find a formula for the angle between two non-parallel planes in space. Also, find a formula for the angle between a line and plane in space which are non-parallel.

To find the angle between two non-parallel planes, simply use the perpendicular vectors for each plane, since they will preserve the angle between planes. We already have a formula (equation 6.7) for the angle between vectors involving the dot product, or one can use the cross product formula (equation 6.12) as well.

As for the angle between a line and plane, note that the sum of (1) the angle between the line and the plane and (2) the angle between the line and the vector perpendicular to the plane, must equal $\frac{\pi}{2}$. Therefore, if you compute (2), which is straight forward, then subtracting that value from $\frac{\pi}{2}$ yields (1).

# Chapter 8

# Independence, Basis, and Dimension for Subspaces of $\mathbb{R}^n$

## 8.1 Subspaces of $\mathbb{R}^n$

1. Determine if each of the following sets define a vector space over some field $\mathbf{F}$.

(a) $\mathbb{V} = \{\langle x, y, 0\rangle \mid x, y \in \mathbb{R}\}$ - yes

(b) $\mathbb{V} = \{\langle x, 1, z\rangle \mid x, z \in \mathbb{Q}\}$ - no, closure properties are not satisfied

(c) $\mathbb{V} = \{P(x) \mid P(x)$ is a polynomial with real coefficients$\}$ - yes

(d) $\mathbb{V} = \{P(x) \mid P(x)$ is a cubic polynomial with complex coefficients$\}$ - no, no vector identity.

(e) $\mathbb{V} = \{\langle x, y, z\rangle \mid x, y, z \geq 0\}$ - no, no additive inverse.

(f) $\mathbb{V} = \mathbb{Q}^{4\times 3}$, which are the rational $4 \times 3$ matrices. - yes, but only if scalars are rational.

(g) $\mathbb{V} = \mathbb{Z}^{3\times 3}$, which are the integer $3 \times 3$ matrices. - yes, but only if scalars are integers.

(h) $\mathbb{V}$ is the set of all polynomials with integer coefficients. - yes, but only if scalars are integers.

2. Prove or disprove that the following vector subspace unions, $\mathbb{U} \cup \mathbb{V}$, are themselves vector subspaces.

(a) $\mathbb{U} = \{\langle 0, y, 0\rangle \mid y \in \mathbb{R}\}, \mathbb{V} = \{\langle 0, 0, z\rangle \mid z \in \mathbb{R}\}$

No, the closure properties are not satisfied, as no vector of the form $\langle 0, y, z\rangle$ can be in the union, but it should be for $\mathbb{U} \cup \mathbb{V}$ to be a vector space.

(b) $\mathbb{U} = \{\langle x, y, 0\rangle, \mid x, y \in \mathbb{R}\}, \mathbb{V} = \{\langle 0, 0, z\rangle \mid z \in \mathbb{R}\}$

No, the same argument from part (a) holds here, except this time, no vector of the form $\langle x, y, z\rangle$ is in the union.

(c) $\mathbb{U} = \{\langle x, y, 0\rangle \mid x, y \in \mathbb{R}\}, \mathbb{V} = \{\langle x, 0, 0\rangle \mid x \in \mathbb{R}\}$

Yes, this is true since $\mathbb{V} \subset \mathbb{U}$, and $\mathbb{U}$ itself is a vector space.

3. Express the sum, $\mathbb{U} + \mathbb{V}$, of the following vector subspaces $\mathbb{U}$ and $\mathbb{V}$ as a single vector subspace.

(a) $\mathbb{U} = \big\{\langle 0, a, 0\rangle, \mid a \in \mathbb{R}\big\}, \mathbb{V} = \big\{\langle 0, 0, b\rangle, \mid b \in \mathbb{R}\big\}$

$\mathbb{U} + \mathbb{V} = \big\{\langle 0, a, b\rangle, \mid a, b \in \mathbb{R}\big\}$

(b) $\mathbb{U} = \big\{\langle a, b, 0\rangle, \mid a, b \in \mathbb{R}\big\}, \mathbb{V} = \big\{\langle 0, 0, c\rangle, \mid c \in \mathbb{R}\big\}$

$\mathbb{U} + \mathbb{V} = \big\{\langle a, b, c\rangle, \mid a, b, c \in \mathbb{R}\big\} = \mathbb{R}^3$

(c) $\mathbb{U} = \big\{\langle a, b, 0\rangle, \mid a, b \in \mathbb{R}\big\}, \mathbb{V} = \big\{\langle a, 0, 0\rangle, \mid a \in \mathbb{R}\big\}$

$\mathbb{U} + \mathbb{V} = \mathbb{U}$

(d) $\mathbb{U} = \big\{\langle a, 0, b, 0, c, d, e, 0\rangle, \mid a, b, c, d, e \in \mathbb{R}\big\}$,
$\mathbb{V} = \big\{\langle 0, a, b, c, 0, 0, d, 0\rangle, \mid a, b, c, d \in \mathbb{R}\big\}$

$\mathbb{U} + \mathbb{V} = \big\{\langle a, b, c, d, e, f, g, 0\rangle, \mid a, b, c, d, e, f, g \in \mathbb{R}\big\}$

(e) $\mathbb{U} = \big\{a\,\overrightarrow{u_1} + b\,\overrightarrow{u_2} + c\,\overrightarrow{u_3}, \mid a, b, c \in \mathbb{R}\big\}$
$\mathbb{V} = \big\{a\,\overrightarrow{v_1} + b\,\overrightarrow{v_2} + c\,\overrightarrow{v_3} + d\,\overrightarrow{v_4}, \mid a, b, c, d \in \mathbb{R}\big\}$,
for fixed vectors $\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{u_3}, \overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}, \overrightarrow{v_4} \in \mathbb{R}^9$ where $\overrightarrow{u_1}, \overrightarrow{u_2} \in \mathbb{V}$.

$$\mathbb{U}+\mathbb{V}=\left\{a\,\overrightarrow{v_1}+b\,\overrightarrow{v_2}+c\,\overrightarrow{v_3}+d\,\overrightarrow{v_4}+e\,\overrightarrow{u_3},\ \middle|\ a,b,c,d,e\in\mathbb{R}\right\}$$

4. Express the solutions to the following homogeneous systems as vector subspaces of $\mathbb{R}^n$, also state the dimension of each solution.

(a) $\begin{aligned}3x-6y+5z&=0\\-x+3y-2z&=0\end{aligned}\ \rightarrow \mathbb{S}=\left\{\left\langle -a,\frac{a}{3},a\right\rangle\ \middle|\ a\in\mathbb{R}\right\}$

(b) $\begin{aligned}x-2y-3z&=0\\-2x+5y-7z&=0\end{aligned}\ \rightarrow \mathbb{S}=\{\langle 29\,a,13\,a,a\rangle \mid a\in\mathbb{R}\}$

(c) $\begin{aligned}w+3x+y+2z&=0\\2w-3x+4y+6z&=0\end{aligned}\ \rightarrow \mathbb{S}=\left\{\left\langle -\frac{15}{2}\,a-b,a,\frac{9}{2}\,a-b,\,b\right\rangle\ \middle|\ a,\,b\in\mathbb{R}\right\}$

(d) $\begin{aligned}-2w+x+y&=0\\2w+5x+y-2z&=0\\-w+x+y+z&=0\end{aligned}\ \rightarrow \mathbb{T}=\left\{\left\langle a,-\frac{3}{2}\,a,\frac{7}{2}\,a,\,-a\right\rangle\ \middle|\ a\in\mathbb{R}\right\}$

Vector subspaces corresponding to (a), (b), and (d) have dimension 1, and (c) has dimension 2.

5. Express the solutions to the following nonhomogeneous systems as vector subspace translates of $\mathbb{R}^n$, also state the dimension of each solution.

(a) $\begin{aligned}3x-6y+5z&=-1\\-x+3y-2z&=5\end{aligned}\ \rightarrow \mathbb{T}=\left\{\left\langle -a,\frac{a}{3},a\right\rangle+\left\langle 9,\frac{14}{3},0\right\rangle\ \middle|\ a\in\mathbb{R}\right\}$

(b) $\begin{aligned}x-2y-3z&=3\\-2x+5y-7z&=2\end{aligned}\ \rightarrow \mathbb{T}=\{\langle 29\,a,13\,a,a\rangle+\langle 19,8,0\rangle \mid a\in\mathbb{R}\}$

(c) $\begin{aligned}w+3x+y+2z&=-1\\2w-3x+4y+6z&=-2\end{aligned}$

$$\rightarrow \mathbb{T}=\left\{\left\langle -\frac{15}{2}\,a-b,a,\frac{9}{2}\,a-b,\,b\right\rangle+\langle -1,0,0,0\rangle\ \middle|\ a,\,b\in\mathbb{R}\right\}$$

( d) $\begin{aligned}-2w+x+y&=1\\2w+5x+y-2z&=3\\-w+x+y+z&=2\end{aligned}\ \rightarrow \mathbb{S}=\left\{\left\langle a,-\frac{3}{2}\,a,\frac{7}{2}\,a,\,-a\right\rangle+\langle 0,1,0,1\rangle\ \middle|\ a\in\mathbb{R}\right\}$

6. Explain why the following sets $\mathbb{S}$ are subspaces, or not, of the appropriate vector space $\mathbb{R}^n$:

(a) $\mathbb{S} = \{\,\langle a+b+5, 2a+7b-1, a-b+4\rangle \mid a, b, \in \mathbb{R}\}$

No. No combinations of $a$ and $b$ will result in the zero vector, which is required for $\mathbb{S}$ to be a subspace.

(b) $\mathbb{S} = \{\,\langle 4a-b, -5a+7b, a-3b, -10a+\pi\, b\rangle \mid a, b, \in \mathbb{R}\}$

Yes, all properties of being a subspace are satisfied.

(c) $\mathbb{S} = \{\,\langle 5, a+3b, 0, a-b, \pi\rangle \mid a, b, \in \mathbb{R}\}$

No. No combinations of $a$ and $b$ will result in the zero vector, which is required for $\mathbb{S}$ to be a subspace.

(d) $\mathbb{S} = \{\,\langle a+b, 0, 2a+7b, 0, a-b, 0, 0\rangle \mid a, b, \in \mathbb{R}\}$

Yes, all properties of being a subspace are satisfied.

7. Show that $\mathbb{U} \cap \mathbb{V}$ is a subspace of $\mathbb{R}^n$ if both $\mathbb{U}$ and $\mathbb{V}$ are subspaces of $\mathbb{R}^n$. Does this generalize to the intersection of any finite number of subspaces?

We need to show that if $\overrightarrow{u}$ and $\overrightarrow{v}$ are elements of $\mathbb{U}\cap\mathbb{V}$, then so is $a\overrightarrow{u}+b\overrightarrow{v}$, for scalars $a$ and $b$. By the definition of intersection, if $\overrightarrow{u} \in \mathbb{U}\cap\mathbb{V}$, then $\overrightarrow{u} \in \mathbb{U}$ and $\overrightarrow{u} \in \mathbb{V}$. Furthermore, since $\mathbb{U}$ and $\mathbb{V}$ are themselves vector spaces, $a\overrightarrow{u} \in \mathbb{U}$ and $a\overrightarrow{u} \in \mathbb{V}$, thus implying that $a\overrightarrow{u} \in \mathbb{U}\cap\mathbb{V}$. A similar argument will show that $b\overrightarrow{v} \in \mathbb{U}\cap\mathbb{V}$, and thus $a\overrightarrow{u}+b\overrightarrow{v} \in \mathbb{U}\cap\mathbb{V}$. Clearly a similar argument holds for an intersection of a finite number of subspaces.

8. Find $\mathbb{U} \cap \mathbb{V}$ for the following vector subspaces.

(a) $\mathbb{U} = \{\langle a, 0, b, 0\rangle \mid a, b \in \mathbb{R}\}$, $\mathbb{V} = \{\langle 0, a, b, c\rangle \mid a, b, c \in \mathbb{R}\}$

$\mathbb{U} \cap \mathbb{V} = \{\langle 0, 0, \alpha, 0\rangle \mid \alpha \in \mathbb{R}\}$

(b) $\mathbb{U} = \{\langle a, 0, b, 0, c, d, e, 0\rangle \mid a, b, c, d, e \in \mathbb{R}\}$
$\mathbb{V} = \{\langle 0, a, b, c, 0, 0, d, 0\rangle \mid a, b, c, d \in \mathbb{R}\}$

$\mathbb{U} \cap \mathbb{V} = \{\langle 0, 0, \alpha, 0, 0, 0, \beta, 0\rangle \mid \alpha, \beta \in \mathbb{R}\}$

(c) $\mathbb{U} = \{\langle a, 0, b, 0, c, d, 0, e\rangle \mid a, b, c, d, e \in \mathbb{R}\}$
$\mathbb{V} = \{\langle 0, a, b, c, 0, 0, d, e\rangle \mid a, b, c, d, e \in \mathbb{R}\}$

$\mathbb{U} \cap \mathbb{V} = \{\langle 0, 0, \alpha, 0, 0, 0, 0, \beta\rangle \mid \alpha, \beta \in \mathbb{R}\}$

9. Let $\mathbb{U}$ be the subspace of $\mathbb{R}^n$ which is the solution space to the homogeneous linear system $A\overrightarrow{x} = \overrightarrow{0}$ and $\mathbb{V}$ be the subspace of $\mathbb{R}^n$ which is the solution space to another homogeneous linear system $B\overrightarrow{x} = \overrightarrow{0}$. What homogeneous linear system has the subspace $\mathbb{U} \cap \mathbb{V}$ as its solution space? Is there a homogeneous linear system which has the subspace $\mathbb{U} + \mathbb{V}$ as its solution space?

One has to be careful here. It may be thought that vectors belonging to $\mathbb{U} \cap \mathbb{V}$ satisfy $(A + B)\overrightarrow{x} = \overrightarrow{0}$, however if $A = -B$, then $\mathbb{U} = \mathbb{V}$, but $A + B = 0$ and thus any vector will satisfy $(A + B)\overrightarrow{x} = \overrightarrow{0}$.

Note however, that any vector in $\mathbb{U} \cap \mathbb{V}$ will satisfy

$$\begin{bmatrix} A \\ B \end{bmatrix} \overrightarrow{x} = \overrightarrow{0}.$$

## 8.2 Independent and Dependent Sets of Vectors in $\mathbb{R}^n$

1. Compute the spanning set $\mathbf{K}$ for the subspace $\mathbb{S}$ corresponding to the solution of each of the following homogeneous linear systems.

(a) $\begin{bmatrix} 1 & -2 & 1 \\ 4 & -2 & 8 \end{bmatrix} \overrightarrow{x} = \overrightarrow{0} \to \mathbf{K} = \left\{ \left\langle -\frac{7}{3}, -\frac{2}{3}, 1 \right\rangle \right\}$

(b) $\begin{bmatrix} 8 & -2 & 2 & 4 \\ -4 & 1 & -1 & -2 \end{bmatrix} \overrightarrow{x} = \overrightarrow{0}$

$\to \mathbf{K} = \left\{ \left\langle \frac{1}{4}, 1, 0, 0 \right\rangle, \left\langle -\frac{1}{4}, 0, 1, 0 \right\rangle, \left\langle -\frac{1}{2}, 0, 0, 1 \right\rangle \right\}$

(c) $\begin{bmatrix} 0 & 6 & 1 & 4 \\ 2 & -2 & -9 & -1 \\ -1 & 5 & 2 & -1 \end{bmatrix} \overrightarrow{x} = \overrightarrow{0} \to \mathbf{K} = \left\{ \left\langle -\frac{223}{38}, -\frac{17}{38}, -\frac{25}{19}, 1 \right\rangle \right\}$

(d) $\begin{bmatrix} 0 & 6 \\ 2 & 2 \\ -1 & 5 \end{bmatrix} \overrightarrow{x} = \overrightarrow{0} \to \mathbf{K} = \{\ \}$

(e) $\begin{bmatrix} -2 & 1 & 4 & 2 \\ 2 & 3 & -5 & 0 \\ 0 & 4 & -2 & 2 \end{bmatrix} \vec{x} = \vec{0} \to \mathbf{K} = \left\{ \left\langle \frac{3}{4}, -\frac{1}{2}, 0, 1 \right\rangle \right\}$

(f) $\begin{bmatrix} 2 & 0 & -4 & 2 & 9 \\ 3 & 1 & -7 & 0 & 8 \end{bmatrix} \vec{x} = \vec{0}$

$\to \mathbf{K} = \left\{ \langle 2, 1, 1, 0, 0 \rangle, \langle -1, 3, 0, 1, 0 \rangle, \left\langle -\frac{9}{2}, \frac{11}{2}, 0, 0, 1 \right\rangle \right\}$

2. For each of the homogeneous systems from problem 1, determine the maximum number of vectors that could possibly be in the spanning set. How does this compare to the actual number of vectors in the spanning set?

(a) max is 3, actual is 1

(b) max is 4, actual is 3

(c) max is 4, actual is 1

(d) max is 2, actual is 0

(e) max is 4, actual is 1

(f) max is 5, actual is 3

3. Determine whether or not each of the following pairs of spanning sets $\mathbf{K}_1$ and $\mathbf{K}_2$ span the same subspace:

(a) $\mathbf{K}_1 = \{\langle 1, 0, 1 \rangle, \langle 0, 1, 1 \rangle\}, \quad \mathbf{K}_2 = \{\langle 2, -1, 1 \rangle, \langle -1, 5, 4 \rangle\} \to$ yes

(b) $\mathbf{K}_1 = \{\langle 1, 0, 1 \rangle, \langle 0, 1, 1 \rangle\}, \quad \mathbf{K}_2 = \{\langle 2, -1, 2 \rangle, \langle -1, 5, 4 \rangle\} \to$ no

(c) $\mathbf{K}_1 = \{\langle 2, 3, -1 \rangle, \langle 3, 3, 1 \rangle\}, \quad \mathbf{K}_2 = \{\langle 1, 0, 2 \rangle, \langle -1, -6, 8 \rangle, \langle 2, 0, 5 \rangle\} \to$ no

(d) $\mathbf{K}_1 = \{\langle -1, 0, 1, 0 \rangle, \langle 2, 1, 2, 2 \rangle\}, \quad \mathbf{K}_2 = \{\langle 1, 1, 3, 2 \rangle, \langle 3, 1, 1, 2 \rangle\} \to$ yes

(e) $\mathbf{K}_1 = \{\langle 2, 3, -4, 1 \rangle, \langle -1, 2, -1, 1 \rangle, \langle 2, 1, -1, 1 \rangle\},$
$\mathbf{K}_2 = \{\langle 1, 3, -2, 2 \rangle, \langle 3, 6, -6, 2 \rangle\} \to$ no

(f) $\mathbf{K}_1 = \{\langle 2, 3, -4, 1 \rangle, \langle -1, 2, -1, 1 \rangle, \langle 2, 1, -1, 1 \rangle\},$
$\mathbf{K}_2 = \{\langle 1, 3, -2, 2 \rangle, \langle 3, 6, -6, 3 \rangle, \langle -5, 9, -8, 2 \rangle\} \to$ yes

4. Prove that given a vector subspace $\mathbb{S}$ of $\mathbb{R}^n$, $\mathbb{S}^\perp$, the set of all vectors orthogonal to every vector of $\mathbb{S}$, is also a vector subspace of $\mathbb{R}^n$.

We need to show that if $\overrightarrow{u}, \overrightarrow{v} \in \mathbb{S}^{\perp}$, then so is $a\overrightarrow{u} + b\overrightarrow{v}$. If we take an arbitrary $\overrightarrow{x} \in \mathbb{S}$, notice that

$$\begin{aligned}(a\overrightarrow{u} + b\overrightarrow{v}) \cdot \overrightarrow{x} &= a\overrightarrow{u} \cdot \overrightarrow{x} + b\overrightarrow{v} \cdot \overrightarrow{x} \\ &= a\,0 + b\,0 \\ &= 0.\end{aligned}$$

5. Construct the spanning sets of the orthogonal subspace $\mathbb{S}^{\perp}$ to the subspaces defined by the following spanning sets.

(a) $\mathbf{K} = \{\langle -2, 3\rangle\}$

$\operatorname{rref}\left(\begin{bmatrix} -2 & 3 \end{bmatrix}\right) = \begin{bmatrix} 1 & -\frac{3}{2} \end{bmatrix} \rightarrow \mathbf{K}^{\perp} = \left\{\langle \frac{3}{2}, 1\rangle\right\}$

(b) $\mathbf{K} = \{\langle -2, 3, 3\rangle\}$

$\operatorname{rref}\left(\begin{bmatrix} -2 & 3 & 3 \end{bmatrix}\right) = \begin{bmatrix} 1 & -\frac{3}{2} & -\frac{3}{2} \end{bmatrix} \rightarrow \mathbf{K}^{\perp} = \left\{\left\langle \frac{3}{2}, 1, 0\right\rangle, \left\langle \frac{3}{2}, 0, 1\right\rangle\right\}$

(c) $\mathbf{K} = \{\langle -2, 3, 3\rangle, \langle 0, 3, -7\rangle\}$

$\operatorname{rref}\left(\begin{bmatrix} -2 & 3 & 3 \\ 0 & 3 & -7 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -\frac{7}{3} \end{bmatrix} \rightarrow \mathbf{K}^{\perp} = \left\{\left\langle 5, \frac{7}{3}, 1\right\rangle\right\}$

6. What is the orthogonal subspace to $\mathbb{R}^n$?

The only vector perpendicular to every vector in $\mathbb{R}^n$ is the zero vector, thus $(\mathbb{R}^n)^{\perp} = \left\{\overrightarrow{0}\right\}$.

7. The following two sets, $\mathbf{K}_1$ and $\mathbf{K}_2$, span the same subspace. Explain what this implies about the vectors of $\mathbf{K}_2$.

$$\mathbf{K}_1 = \{\langle 1, 0, 1\rangle, \langle 0, 1, 1\rangle\}, \quad \mathbf{K}_2 = \{\langle 2, -1, 1\rangle, \langle -1, 5, 4\rangle, \langle 1, 4, 5\rangle\}$$

The vectors of $\mathbf{K}_2$ are not linearly independent. Also, each one of the vectors from $\mathbf{K}_2$ can be expressed as a linear combination of vectors from $\mathbf{K}_1$.

8. Let $\mathbb{S}$ be a subspace of $\mathbb{R}^n$. What is $\mathbb{S} + \mathbb{S}^{\perp}$? Explain your answer.

$\mathbb{S} + \mathbb{S}^{\perp} = \mathbb{R}^n$, since if there are $k$ linearly independent vectors in $\mathbb{S}$, there must be $n - k$ directions perpendicular to all of these vectors, which implies that $\mathbb{S}^{\perp}$ would have dimension $n - k$. Adding these dimensions together gives dimension $n$, which is the same as that of $\mathbb{R}^n$.

9. Let $\mathbb{S}$ be a subspace of $\mathbb{R}^n$. What is $\mathbb{S} \cap \mathbb{S}^\perp$? Explain your answer.

$\mathbb{S} \cap \mathbb{S}^\perp = \overrightarrow{0}$ since the only vector they can have in common is one that is perpendicular to all in both, this is the zero vector only.

10. Let $\mathbb{S}$ be a subspace of $\mathbb{R}^n$. What is the dimension of $\mathbb{S}^\perp$? Explain your answer.

This was already described in problem 8. If $\mathbb{S}$ has dimension $k$, then $\mathbb{S}^\perp$ has dimension $n - k$.

11. Find the orthogonal complement $\mathbb{S}^\perp$ for the following subspaces $\mathbb{S}$, and give their dimensions:

(a) $\mathbb{S} = \left\{ \langle 0, 0, a, b, 0, c, 0 \rangle \,\middle|\, a, b, c \in \mathbb{R} \right\}$

$\mathbb{S}^\perp = \left\{ \langle d, e, 0, 0, f, 0, g \rangle \,\middle|\, d, e, f, g \in \mathbb{R} \right\}$

(b) $\mathbb{S} = \left\{ \langle a, 0, b, c, d, 0, 0, e \rangle \,\middle|\, a, b, c, d, e \in \mathbb{R} \right\}$

$\mathbb{S}^\perp = \left\{ \langle 0, f, 0, 0, 0, g, h, 0 \rangle \,\middle|\, f, g, h \in \mathbb{R} \right\}$

## 8.3 Basis and Dimension for Subspaces of $\mathbb{R}^n$

1. Determine which of the following sets are bases for $\mathbb{R}^2$:

(a) $\{\langle 2, 3 \rangle, \langle 2, 1 \rangle\} \to \text{rref}\left( \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to$ basis

(b) $\{\langle -2, 3 \rangle, \langle -3, 1 \rangle\} \to \text{rref}\left( \begin{bmatrix} -2 & 3 \\ -3 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to$ basis

(c) $\{\langle 1, -2 \rangle, \langle -3, 6 \rangle\} \to \text{rref}\left( \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix} \right) = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \to$ not a basis

(d) $\{\langle 0, 2 \rangle, \langle 1, 4 \rangle\} \to \text{rref}\left( \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to$ basis

2. Determine which of the following sets are bases for $\mathbb{R}^3$:

(a) $\{\langle 2,3,1\rangle, \langle 0,2,1\rangle, \langle -1,2,1\rangle\} \rightarrow \text{rref}\left(\begin{bmatrix} 2 & 3 & 1 \\ 0 & 2 & 1 \\ -1 & 2 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\rightarrow$ basis

(b) $\{\langle -2,0,1\rangle, \langle 0,2,0\rangle, \langle 0,0,5\rangle\} \rightarrow \text{rref}\left(\begin{bmatrix} -2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\rightarrow$ basis

(c) $\{\langle 1,1,0\rangle, \langle 0,1,1\rangle, \langle 1,0,1\rangle\} \rightarrow \text{rref}\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\rightarrow$ basis

(d) $\{\langle 1,1,0\rangle, \langle 0,1,1\rangle, \langle 1,0,1\rangle\} \rightarrow \text{rref}\left(\begin{bmatrix} 3 & -2 & 2 \\ 1 & -1 & 0 \\ -5 & 3 & -4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\rightarrow$ not a basis

3. Compute the row rank of the following matrices:

(a) $$\text{rank}\left(\begin{bmatrix} 1 & 3 & 1 \\ -2 & 1 & -2 \\ 0 & 8 & 1 \end{bmatrix}\right) = \text{rank}\left(\text{rref}\left(\begin{bmatrix} 1 & 3 & 1 \\ -2 & 1 & -2 \\ 0 & 8 & 1 \end{bmatrix}\right)\right) = \text{rank}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$\rightarrow$ row rank is three

(b) $$\text{rank}\left(\begin{bmatrix} -3 & -4 & 3 \\ 4 & 1 & 2 \\ 5 & -2 & 7 \end{bmatrix}\right) = \text{rank}\left(\text{rref}\left(\begin{bmatrix} -3 & -4 & 3 \\ 4 & 1 & 2 \\ 5 & -2 & 7 \end{bmatrix}\right)\right) = \text{rank}\left(\begin{bmatrix} 1 & 0 & \frac{11}{13} \\ 0 & 1 & -\frac{18}{13} \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$\rightarrow$ row rank is two

(c) $\text{rank}\left(\begin{bmatrix} -3 & -4 & 3 \\ 4 & 1 & 2 \\ 5 & -2 & 7 \\ 3 & -2 & 1 \end{bmatrix}\right) = \text{rank}\left(\text{rref}\left(\begin{bmatrix} -3 & -4 & 3 \\ 4 & 1 & 2 \\ 5 & -2 & 7 \\ 3 & -2 & 1 \end{bmatrix}\right)\right)$

$= \text{rank}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right)$

$\rightarrow$ row rank is three

(d) $\text{rank}\left(\begin{bmatrix} -3 & -4 & 3 \\ 2 & 2 & 10 \\ 5 & -2 & 7 \\ 8 & -6 & 4 \end{bmatrix}\right) = \text{rank}\left(\text{rref}\left(\begin{bmatrix} -3 & -4 & 3 \\ 2 & 2 & 10 \\ 5 & -2 & 7 \\ 8 & -6 & 4 \end{bmatrix}\right)\right)$

$= \text{rank}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right)$

$\rightarrow$ row rank is three

(e) $\text{rank}\left(\begin{bmatrix} -3 & -4 \\ 4 & 1 \\ 5 & -2 \end{bmatrix}\right) = \text{rank}\left(\text{rref}\left(\begin{bmatrix} -3 & -4 \\ 4 & 1 \\ 5 & -2 \end{bmatrix}\right)\right)$

$= \text{rank}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$

$\rightarrow$ row rank is two

(f) $\text{rank}\left(\begin{bmatrix} -3 & -4 & 3 & 0 \\ 4 & 1 & 2 & -1 \\ 5 & -2 & 7 & 3 \end{bmatrix}\right) = \text{rank}\left(\text{rref}\left(\begin{bmatrix} -3 & -4 & 3 & 0 \\ 4 & 1 & 2 & -1 \\ 5 & -2 & 7 & 3 \end{bmatrix}\right)\right)$

$= \text{rank}\left(\begin{bmatrix} 1 & 0 & \frac{11}{13} & 0 \\ 0 & 1 & -\frac{18}{13} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)$

$\rightarrow$ row rank is three

4. Determine the maximum possible row rank for each of the matrices from problem 3.

(a) 3 (b) 3 (c) 3 (d) 3 (e) 2 (f) 3

5. Construct a basis for each of the following subspaces of $\mathbb{R}^n$.

(a) the set of all vectors in $\mathbb{R}^3$ of the form $\langle a, b, a\rangle$

$\mathbf{B} = \{\langle 1,0,1\rangle,\ \langle 0,1,0\rangle\}$

(b) the set of all vectors in $\mathbb{R}^4$ of the form $\langle a, b, -a, -b\rangle$

$\mathbf{B} = \{\langle 1,0,-1,0\rangle,\ \langle 0,1,0,-1\rangle\}$

(c) the set of all vectors in $\mathbb{R}^3$ of the form $\langle a, b, a-b\rangle$

$\mathbf{B} = \{\langle 1,0,1\rangle,\ \langle 0,1,-1\rangle\}$

(d) the set of all vectors in $\mathbb{R}^4$ of the form $\langle a, 2b, a-3b, 2a+3b+c\rangle$

$\mathbf{B} = \{\langle 1,0,1,2\rangle,\ \langle 0,2,-3,3\rangle,\ \langle 0,0,0,1\rangle\}$

6. Let $C = (A\,|\,0)$ be the augmented matrix for the homogeneous linear system $A\overrightarrow{x} = \overrightarrow{0}$, where $A \in \mathbb{R}^{m\times n}$. Now apply *rref* to this matrix $C$ in order to read off the solutions $\overrightarrow{x}$ to this system. You will get from rref($C$) that the solutions $\overrightarrow{x}$ are of the form

$$\overrightarrow{x} = x_{k_1}\overrightarrow{u_1} + x_{k_2}\overrightarrow{u_2} + \cdots + x_{k_p}\overrightarrow{u_p}$$

where $x_{k_1}, x_{k_1}, \ldots, x_{k_p}$ are arbitrary solution variables from $\overrightarrow{x}$, and $\overrightarrow{u_1}$, $\overrightarrow{u_2}, \ldots$, $\overrightarrow{u_p} \in \mathbb{R}^n$ are $p$ fixed solutions. Are the column vectors $\overrightarrow{u_1}, \overrightarrow{u_2}, \ldots, \overrightarrow{u_p}$ automatically a basis of the subspace $\mathbb{S}$ of $\mathbb{R}^n$ consisting of all the solutions $\overrightarrow{x}$ to the homogeneous system $A\overrightarrow{x} = \overrightarrow{0}$? Explain your answer in detail.

Indeed the answer is yes! To see this, consider the fact that $A\overrightarrow{u_j} = \overrightarrow{0}$ for $1 \leq j \leq p$. Therefore, any linear combination of the $\overrightarrow{u_j}$'s will also be a solution to $A\overrightarrow{x} = \overrightarrow{0}$, i.e. we can take our solution to be of the form given above and thus

$$\begin{aligned} A\overrightarrow{x} &= A\left(x_{k_1}\overrightarrow{u_1} + x_{k_2}\overrightarrow{u_2} + \cdots + x_{k_p}\overrightarrow{u_p}\right) \\ &= x_{k_1}A\overrightarrow{u_1} + x_{k_2}A\overrightarrow{u_2} + \cdots + x_{k_p}A\overrightarrow{u_p} \end{aligned}$$

Since the $x_k$'s are arbitrary, the $\overrightarrow{u_j}$'s form a subspace of $\mathbb{R}^n$. Of course, the bigger question to ask is: Does this subspace contain all solutions to $A\overrightarrow{x} = \overrightarrow{0}$? If it does not, then there is another vector, call it $\overrightarrow{u_{p+1}}$, which we missed the first time around. There can be at most $n$ of these vectors (why?), so we can conclude that we do cover all possible solutions to $A\overrightarrow{x} = \overrightarrow{0}$, and the solutions form a subspace of $\mathbb{R}^n$.

7. (a) Let $\mathbb{S}$ be a subspace of $\mathbb{R}^n$ with a basis $\mathbf{B} = \left\{\overrightarrow{u_1}, \overrightarrow{u_2}, \ldots, \overrightarrow{u_p}\right\}$. Explain in detail how you can find a basis for the orthogonal complement $\mathbb{S}^\perp$.

We can construct a matrix $B$ whose rows are the given vectors. Then, solving the equation $B\overrightarrow{x} = \overrightarrow{0}$ will give us a basis for $\mathbb{S}^\perp$. Remember, that when performing matrix multiplication, you multiply the row of the left matrix by the column of the right matrix, which is equivalent to the dot product. So when the dot product is zero, the vectors are orthogonal.

(b) What is the dimension of $\mathbb{S}^\perp$ in terms of $n$ and the dimension $p$ of $\mathbb{S}$ Explain why this is true.

The dimension of $\mathbb{S}^\perp$ is $n - p$, since $\mathbb{S} + \mathbb{S}^\perp = \mathbb{R}^n$, which has dimension $n$.

8. For the following subspaces $\mathbb{S}$ of $\mathbb{R}^n$, find a basis for both $\mathbb{S}$ and its orthogonal complement $\mathbb{S}^\perp$, giving the dimension of each.

(a) $\mathbb{S} = \left\{\langle 0, 0, a, b, 0, c, 0\rangle \,\middle|\, a, b, c \in \mathbb{R}^7\right\}$

$\mathbb{S}^\perp = \left\{\langle d, e, 0, 0, f, 0, g\rangle \,\middle|\, d, e, f, g \in \mathbb{R}\right\}$

(b) $\mathbb{S} = \left\{\langle a, 0, b, c, d, 0, 0, e\rangle \,\middle|\, a, b, c, d, e \in \mathbb{R}^7\right\}$

$\mathbb{S}^\perp = \left\{\langle 0, f, 0, 0, 0, g, h, 0\rangle \,\middle|\, f, g, h \in \mathbb{R}\right\}$

## 8.4 Vector Projection onto a Subspace of $\mathbb{R}^n$

1. Verify that if $\mathbb{S}$ is a one-dimensional subspace of $\mathbb{R}^n$ spanned by the single vector $\overrightarrow{w}$, then the formula for $\text{proj}_{\mathbb{S}}(\overrightarrow{v})$ reduces to the formula given by $\text{proj}_{\overrightarrow{w}}(\overrightarrow{v})$.

Equation 8.31 is a good place to start. So in this case $C = \overrightarrow{w}$ where $\overrightarrow{w}$ is treated as an $n \times 1$ matrix. Thus, we have

$$\text{proj}_{\mathbb{S}}(\overrightarrow{v}) = \left(\overrightarrow{w}\left(\overrightarrow{w}^T\overrightarrow{w}\right)^{-1}\overrightarrow{w}^T\overrightarrow{v}\right)^T$$

But notice that $\overrightarrow{w}^T\overrightarrow{w} = \overrightarrow{w} \cdot \overrightarrow{w} = |\overrightarrow{w}|^2$, and therefore, we get

$$\begin{aligned}
\text{proj}_{\mathbb{S}}(\overrightarrow{v}) &= \left(\overrightarrow{w}\left(\overrightarrow{w}^T\overrightarrow{w}\right)^{-1}\overrightarrow{w}^T\overrightarrow{v}\right)^T \\
&= \frac{1}{|\overrightarrow{w}|^2}\left(\overrightarrow{w}\,\overrightarrow{w}^T\overrightarrow{v}\right)^T \\
&= \frac{1}{|\overrightarrow{w}|^2}\left(\overrightarrow{v}^T\overrightarrow{w}\,\overrightarrow{w}^T\right) \\
&= \frac{\overrightarrow{v}\cdot\overrightarrow{w}}{|\overrightarrow{w}|^2}\overrightarrow{w}^T
\end{aligned}$$

which is the definition of $\text{proj}_{\overrightarrow{w}}(\overrightarrow{v})$.

2. Given the basis $\mathbf{B} = \{\overrightarrow{w_1}, \overrightarrow{w_2}, \ldots, \overrightarrow{w_k}\}$ for a subspace of $\mathbb{R}^n$, prove that the matrix $W \in \mathbb{R}^{k\times k}$, defined by $W_{i,j} = \overrightarrow{w_i} \cdot \overrightarrow{w_j}$, satisfies $W^T = W$.

This is obviously true since the vectors are real and thus $\overrightarrow{w_j} \cdot \overrightarrow{w_i} = \overrightarrow{w_i} \cdot \overrightarrow{w_j}$, which shows that $W_{j,i} = W_{i,j}$ and thus $W^T = W$.

3. Given the basis $\mathbf{B} = \{\overrightarrow{w_1}, \overrightarrow{w_2}, \ldots, \overrightarrow{w_k}\}$ for a subspace of $\mathbb{C}^n$, and $W \in \mathbb{C}^{k\times k}$ defined by $W_{i,j} = \overrightarrow{w_i} \cdot \overrightarrow{w_j}$, determine a relationship between $W$ and $W^T$.

Using the same argument as the last problem, we know that $\overrightarrow{w_j} \cdot \overrightarrow{w_i} = \overline{\overrightarrow{w_i} \cdot \overrightarrow{w_j}}$, by definition of the complex dot product. This implies that $W_{j,i} = \overline{W_{i,j}}$ and thus $W^T = \overline{W}$.

4. Project the following vectors on the subspace of $\mathbb{R}^3$ generated by the basis $\{\langle 1,1,0\rangle, \langle 0,0,1\rangle\}$.

(a) $\text{proj}_{\mathbb{S}}(\langle -1,2,1\rangle) = \left\langle \frac{1}{2}, \frac{1}{2}, 1\right\rangle$

(b) $\text{proj}_{\mathbb{S}}(\langle -1,2,0\rangle) = \left\langle \frac{1}{2}, \frac{1}{2}, 0\right\rangle$

(c) $\text{proj}_{\mathbb{S}}(\langle 0,2,3\rangle) = \langle 1,1,3\rangle$

5. Modify the vector projection formula given in equation (8.31) for when the vectors are complex valued.

All that is required is to replace $A^T$ with $\overline{A}^T$. This is usually denoted $A^*$ and is called the *Hermitian transpose* of the matrix $A$. Thus

$$\text{proj}_{\mathbb{S}}(\overrightarrow{v}) = A^*\overrightarrow{v}$$

6. Project the vector $\overrightarrow{v} = \langle -2+i, 6-i, 3+2i\rangle$ onto the subspace of $\mathbb{C}^3$ generated by the basis $\{\langle i,1,1+i\rangle, \langle 0,i,1-i\rangle\}$.

$\text{proj}_{\mathbb{S}}(\langle -2+i, 6-i, 3+2i\rangle) = \left\langle \frac{37}{11}\, i, \frac{53}{11}, \frac{21}{11} + \frac{21}{11}\, i\right\rangle$

Note that $\overrightarrow{v} - \text{proj}_{\mathbb{S}}(\langle -2+i, 6-i, 3+2i\rangle) = \left\langle -2 - \frac{26}{11}\, i, \frac{13}{11} - i, \frac{12}{11} + \frac{1}{11}\, i\right\rangle$, and

$$\left\langle -2 - \frac{26}{11}\, i, \frac{13}{11} - i, \frac{12}{11} + \frac{1}{11}\, i\right\rangle \cdot \langle i, 1, 1+i\rangle = 0$$
$$\left\langle -2 - \frac{26}{11}\, i, \frac{13}{11} - i, \frac{12}{11} + \frac{1}{11}\, i\right\rangle \cdot \langle 0, i, 1-i\rangle = 0$$

Do not forget to take the conjugate of the second vector in each dot product!

7. Prove that the dot product matrix $W$, when considering a set of orthogonal vectors, is diagonal. Furthermore, what do the values on the diagonal correspond to?

Remember that $W_{i,j} = \overrightarrow{w_i} \cdot \overrightarrow{w_j}$. If $i \neq j$ then $\overrightarrow{w_i} \neq \overrightarrow{w_j}$ and therefore $\overrightarrow{w_i} \cdot \overrightarrow{w_j} = 0$. This implies that any off-diagonal entry is zero. The diagonal entries are given by $\overrightarrow{w_i} \neq \overrightarrow{w_i} = |\overrightarrow{w_i}|^2$. This the matrix $W$ is diagonal with diagonal entries equal to the magnitude squared of each vector. In the case of an orthonormal set of vectors, we end up with the identity matrix.

8. Determine if each of the following sets of vectors constitute an orthogonal set:

(a) $\{\langle 1, 0, -1\rangle, \langle 0, 1, 0\rangle, \langle 1, 0, 1\rangle\}$

$$W = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow \text{ set is orthogonal}$$

(b) $\{\langle 1, 1, 1\rangle, \langle 2, -2, 2\rangle, \langle 6, 0, -6\rangle\}$

$$W = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 12 & 0 \\ 0 & 0 & 72 \end{bmatrix} \rightarrow \text{ set is not orthogonal}$$

(c) $\{\langle 1, 2, 1\rangle, \langle 2, -2, 2\rangle, \langle 6, 0, -6\rangle\}$

$$W = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 72 \end{bmatrix} \rightarrow \text{ set is orthogonal}$$

(d) $\{\langle 1, 1, -1, 1\rangle, \langle 0, 1, 0, -1\rangle, \langle -2, 1, 2, 2\rangle\}$

$$W = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 13 \end{bmatrix} \rightarrow \text{ set is not orthogonal}$$

(e) $\{\langle 1,1,-1,1\rangle, \langle 0,1,0,-1\rangle, \langle -3,2,1,2\rangle\}$

$$W = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 18 \end{bmatrix} \rightarrow \text{ set is orthogonal}$$

9. Convert each set from problem 8 that was determined to be an orthogonal set into an orthonormal basis.

All we have to do is make each vector unit length from parts (a), (c), and (e). Notice the length squared of each vector is given along the diagonal, so the answers are pretty simple.

(a) $\left\{\frac{1}{\sqrt{2}}\langle 1,0,-1\rangle, \langle 0,1,0\rangle, \frac{1}{\sqrt{2}}\langle 1,0,1\rangle\right\}$

(c) $\left\{\frac{1}{\sqrt{6}}\langle 1,2,1\rangle, \frac{1}{\sqrt{12}}\langle 2,-2,2\rangle, \frac{1}{\sqrt{72}}\langle 6,0,-6\rangle\right\}$

(c) $\left\{\frac{1}{2}\langle 1,1,-1,1\rangle, \frac{1}{\sqrt{2}}\langle 0,1,0,-1\rangle, \frac{1}{\sqrt{18}}\langle -3,2,1,2\rangle\right\}$

10. Compute the distance from the plane spanned by the vectors $\{\langle 1,0,1\rangle, \langle 1,2,-1\rangle\}$ to the point $(8,8,1)$.

We simply need to compute $|\overrightarrow{v} - \text{proj}_{\mathbb{S}}(\overrightarrow{v})|$, where $\mathbb{S}$ is the subspace spanned by the two vectors which create the plane. We apply the subspace vector projection formula to get $|\text{proj}_{\mathbb{S}}(\overrightarrow{v})| = \left\langle \frac{25}{3}, \frac{23}{3}, \frac{2}{3}\right\rangle$. Then

$$|\overrightarrow{v} - \text{proj}_{\mathbb{S}}(\overrightarrow{v})| = \left|\left\langle -\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\rangle\right| = \frac{1}{\sqrt{3}}$$

11. Prove Theorem 8.4.1.

The proof is straight forward. Notice that if we assume that **B** is an orthonormal basis, then $W$ is the identity matrix. Using this fact in equation (8.31) gives $\text{proj}_{\mathbb{S}}(\overrightarrow{v}) = V^T C^T$, where $V_i = \overrightarrow{w_i} \cdot \overrightarrow{v}$. This yields

$$\text{proj}_{\mathbb{S}}(\overrightarrow{v}) = (\overrightarrow{w_1} \cdot \overrightarrow{v})\,\overrightarrow{w_1} + (\overrightarrow{w_2} \cdot \overrightarrow{v})\,\overrightarrow{w_2} + \cdots + (\overrightarrow{w_k} \cdot \overrightarrow{v})\,\overrightarrow{w_k}$$

Notice though, that since $\overrightarrow{v} \in \mathbb{S}$, $\text{proj}_{\mathbb{S}}(\overrightarrow{v}) = \overrightarrow{v}$, which gives

$$\overrightarrow{v} = (\overrightarrow{w_1} \cdot \overrightarrow{v})\,\overrightarrow{w_1} + (\overrightarrow{w_2} \cdot \overrightarrow{v})\,\overrightarrow{w_2} + \cdots + (\overrightarrow{w_k} \cdot \overrightarrow{v})\,\overrightarrow{w_k}.$$

A similar argument holds conversely.

12. Is vector projection linear? That is, for a subspace $\mathbb{S}$ of $\mathbb{R}^n$ and two vectors $\overrightarrow{u}, \overrightarrow{v} \in \mathbb{R}^n$ and scalar $c$, determine if the following two properties hold. If the properties do not hold, give an example.

(a) $\text{proj}_{\mathbb{S}}(c\overrightarrow{v}) = c\,\text{proj}_{\mathbb{S}}(\overrightarrow{v})$

(b) $\text{proj}_{\mathbb{S}}(\overrightarrow{u} + \overrightarrow{v}) = \text{proj}_{\mathbb{S}}(\overrightarrow{u}) + \text{proj}_{\mathbb{S}}(\overrightarrow{v})$

By construction of the projection (which is the result of matrix multiplication), and the fact that the dot product is also linear, both properties hold. Refer to equations (8.31) and (8.32) for verification of this fact.

13. Let $\mathbb{S}$ be a subspace of $\mathbb{R}^n$ with basis $\mathbf{B} = \{\overrightarrow{w_1}, \overrightarrow{w_2}, \dots, \overrightarrow{w_k}\}$ and let $\overrightarrow{v} \in \mathbb{R}^n$. Determine the conditions on $\overrightarrow{v}$ under which the following condition holds:

$$\text{proj}_{\mathbb{S}}(\overrightarrow{v}) = \text{proj}_{\overrightarrow{w_1}}(\overrightarrow{v}) + \text{proj}_{\overrightarrow{w_2}}(\overrightarrow{v}) + \cdots + \text{proj}_{\overrightarrow{w_k}}(\overrightarrow{v})$$

Theorem 8.4.1 states this property, so if the basis is orthonormal, we get the desired result.

14. What is $\text{proj}_{\mathbb{S}}\left(\text{proj}_{\mathbb{S}}(\overrightarrow{v})\right)$?

First, notice that $\text{proj}_{\mathbb{S}}(\overrightarrow{v}) \in \mathbb{S}$, as a result, performing the projection once more results in no new change. Thus $\text{proj}_{\mathbb{S}}\left(\text{proj}_{\mathbb{S}}(\overrightarrow{v})\right) = \text{proj}_{\mathbb{S}}(\overrightarrow{v})$. This is a standard property of a projection operator (call it $P$), as they satisfy the property $P \circ P = P$, where $\circ$ is the composition of operators.

15. Let $\mathbb{S}$ be a subspace of $\mathbb{R}^n$ where $\mathbf{B} = \{\overrightarrow{w_1}, \overrightarrow{w_2}, \dots, \overrightarrow{w_k}\}$ is a basis of $\mathbb{S}$. Can we extend $\mathbf{B}$ into a full basis of all of $\mathbb{R}^n$? *Hint: Consider a basis of the orthogonal complement* $\mathbb{S}^{\perp}$.

From section 8.3 homework problems, we know that if we can find the orthogonal subspace $\mathbb{S}^{\perp}$, to $\mathbb{S}$ by solving $B\overrightarrow{x} = \overrightarrow{0}$, where $B$ has as its rows the vectors of $\mathbf{B}$. Each of these vectors which satisfy $B\overrightarrow{x} = \overrightarrow{0}$ are orthogonal to $\mathbb{S}$, and are linearly independent, thus completing a set of basis vectors for $\mathbb{R}^n$.

16. (a) Let $\mathbb{S}$ be a subspace of $\mathbb{R}^n$ with basis $\mathbf{B} = \{\overrightarrow{w_1}, \overrightarrow{w_2}, \overrightarrow{w_3}\}$. Show that the set $\mathbf{B_1} = \{\overrightarrow{q_1}, \overrightarrow{q_2}, \overrightarrow{q_3}\}$ is an orthogonal basis of $\mathbb{S}$, where

$$\overrightarrow{q_1} = \overrightarrow{w_1},\ \overrightarrow{q_2} = \overrightarrow{w_2} - \text{proj}_{\overrightarrow{q_1}}(\overrightarrow{w_2}),\ \overrightarrow{q_3} = \overrightarrow{w_3} - \text{proj}_{\overrightarrow{q_1}}(\overrightarrow{w_3}) - \text{proj}_{\overrightarrow{q_2}}(\overrightarrow{w_3})$$

(b) Generalize part (a) to a subspace $\mathbb{S}$ of any dimension.

See the next section on the Gram-Schmidt Orthonormalization process for answers to this problem.

17. Let $\mathbb{U}$ and $\mathbb{V}$ be two subspaces of $\mathbb{R}^n$. For $\overrightarrow{w} \in \mathbb{R}^n$, find conditions on $\mathbb{U}$ and $\mathbb{V}$ such that

$$\text{proj}_{\mathbb{U}+\mathbb{V}}(\overrightarrow{w}) = \text{proj}_{\mathbb{U}}(\overrightarrow{w}) + \text{proj}_{\mathbb{V}}(\overrightarrow{w})$$

If the basis vectors for $\mathbb{U}$ and $\mathbb{V}$ are mutually orthogonal, the above formula holds.

18. (*Fourier Series*) Let $\mathbb{V}$ be the real vector space of all continuous functions $f : [0, 2\pi] \to \mathbb{R}$. Define the the dot product of two elements $f(x)$ and $g(x)$ of $\mathbb{V}$ by

$$f(x) \cdot g(x) = \int_0^{2\pi} f(x)g(x)\,dx$$

Compute the vector projection $\text{proj}_{\mathbb{S}}(\overrightarrow{v})$, for $\overrightarrow{v} = e^x \in \mathbb{V}$, and $\mathbb{S}$, the subspace of $\mathbb{V}$, having basis

$$\mathbf{B} = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(x), \frac{1}{\sqrt{\pi}}\sin(x), \frac{1}{\sqrt{\pi}}\cos(2x), \frac{1}{\sqrt{\pi}}\sin(2x) \right\}$$

(You should first check if $\mathbf{B}$ is orthonormal.) Now graph together both $\overrightarrow{v}$ and $\text{proj}_{\mathbb{S}}(\overrightarrow{v})$. Is $\text{proj}_{\mathbb{S}}(\overrightarrow{v})$ a reasonable approximation of $\overrightarrow{v}$, and how can you make $\text{proj}_{\mathbb{S}}(\overrightarrow{v})$ into a better approximation of $\overrightarrow{v}$?

It is easy enough to check that $\mathbf{B}$ is an orthonormal set, and we will use equation (8.33) as a result of this fact.

$$\begin{aligned}
\text{proj}_{\mathbb{S}}(e^x) = &\int_0^{2\pi} \frac{1}{\sqrt{2\pi}} e^x\,dx \frac{1}{\sqrt{2\pi}} + \int_0^{2\pi} \frac{1}{\sqrt{\pi}} \cos(x)\, e^x\,dx \frac{1}{\sqrt{\pi}} \cos(x) \\
&+ \int_0^{2\pi} \frac{1}{\sqrt{\pi}} \sin(x)\, e^x\,dx \frac{1}{\sqrt{\pi}} \sin(x) \\
&+ \int_0^{2\pi} \frac{1}{\sqrt{\pi}} \cos(2x)\, e^x\,dx \frac{1}{\sqrt{\pi}} \cos(2x) \\
&+ \int_0^{2\pi} \frac{1}{\sqrt{\pi}} \sin(2x)\, e^x\,dx \frac{1}{\sqrt{\pi}} \sin(2x)
\end{aligned}$$

Using the above formula, we have

$$\begin{aligned}
\text{proj}_{\mathbb{S}}(e^x) = &\frac{e^{2\pi}-1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} + \frac{e^{2\pi}-1}{2\sqrt{\pi}} \frac{1}{\sqrt{\pi}} \cos(x) \\
&- \frac{e^{2\pi}-1}{2\sqrt{\pi}} \frac{1}{\sqrt{\pi}} \sin(x) + \frac{e^{2\pi}-1}{5\sqrt{\pi}} \frac{1}{\sqrt{\pi}} \cos(2x) \\
&- \frac{2e^{2\pi}-2}{5\sqrt{\pi}} \frac{1}{\sqrt{\pi}} \sin(2x)
\end{aligned}$$

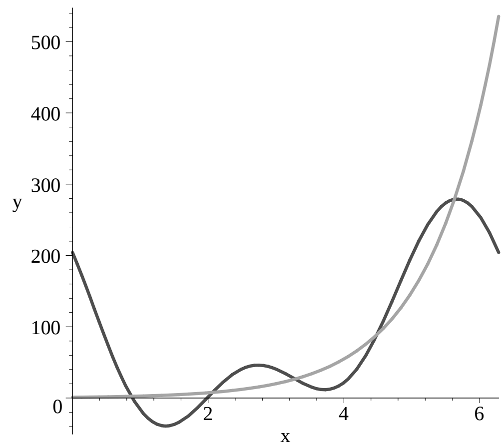


Figure 8.1: The exponential function and its projection onto the given basis.

Notice that the approximation above is not that great, mainly due to the periodicity of the basis functions. To get a better approximation, one can simply add more basis functions, however there will be problems (except perhaps in the limiting sense) of the accuracy at the endpoints $x = 0$ and $x = 2\pi$.

## 8.5 The Gram-Schmidt Orthonormalization Process

1. Explain what happens if one attempts to apply the Gram-Schmidt orthonormalization process to a set of vectors that is linearly dependent. It may be easiest to assume that $\mathbf{K} = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_k}, \overrightarrow{v_{k+1}}\}$ and that

$$\overrightarrow{v_{k+1}} = \sum_{j=1}^{k} a_k \overrightarrow{v_k}$$

for some scalars $a_k$, $1 \leq j \leq k$, of which at least one is nonzero.

The vector which is a linear combination of the other vectors will become the zero vector when the Gram-Schmidt orthonormalization process is applied.

2. What happens when one applies the Gram-Schmidt orthonormalization process to a set of vectors that are already mutually orthogonal?

The process simply normalizes each vector.

3. Convert each of the following sets of vectors to an orthonormal set of vectors:

(a) $\mathbf{K}_1 = \{\langle -2, 3\rangle, \langle 6, 1\rangle\}$

$\mathbf{P}_1 = \left\{\left\langle -\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right\rangle, \left\langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\right\rangle\right\}$

(b) $\mathbf{K}_2 = \{\langle 1, 0, 1\rangle, \langle 0, -1, 1\rangle\}$

$\mathbf{P}_2 = \left\{\left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\rangle, \left\langle -\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right\rangle\right\}$

(c) $\mathbf{K}_3 = \{\langle 1, 0, 1, 1\rangle, \langle 0, -1, 2, 1\rangle, \langle 3, 1, 0, -2\rangle\}$

$\mathbf{P}_3 = \left\{\left\langle \frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\rangle, \left\langle -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0\right\rangle, \left\langle \frac{4}{5\sqrt{3}}, -\frac{1}{5\sqrt{3}}, \frac{3}{5\sqrt{3}}, -\frac{7}{5\sqrt{3}}, 0\right\rangle\right\}$

(d) $\mathbf{K}_4 = \{\langle 1, 1, 0, 1\rangle, \langle 2, 1, -1, 1\rangle, \langle -2, -1, 1, 0\rangle\}$

$$\mathbf{P}_4 = \left\{\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}\right\rangle, \left\langle \frac{2}{\sqrt{15}}, -\frac{1}{\sqrt{15}}, -\frac{3}{\sqrt{15}}, -\frac{1}{\sqrt{15}}\right\rangle,\right.$$
$$\left.\left\langle -\frac{1}{\sqrt{15}}, -\frac{2}{\sqrt{15}}, -\frac{1}{\sqrt{15}}, \frac{3}{\sqrt{15}}, 0\right\rangle\right\}$$

4. Project each of the following vectors onto the corresponding orthonormal basis found in problem 3.

For each of these parts, we use equation (8.33).

(a) $\langle 1, 1, 1\rangle$ onto $\mathbf{K}_2$

$\text{proj}_{\mathbf{K}_2}(\langle 1, 1, 1\rangle) = \left\langle \frac{4}{3}, \frac{2}{3}, \frac{2}{3}\right\rangle$

(b) $\langle 3, 4, -2\rangle$ onto $\mathbf{K}_2$

$\text{proj}_{\mathbf{K}_2}(\langle 3, 4, -2\rangle) = \left\langle \frac{8}{3}, \frac{13}{3}, -\frac{5}{3}\right\rangle$

(c) $\langle 4, -5, -3\rangle$ onto $\mathbf{K}_2$

$\text{proj}_{\mathbf{K}_2}(\langle 4, -5, -3\rangle) = \langle 0, -1, 1\rangle$

(d) $\langle 2, 1, 2, 3\rangle$ onto $\mathbf{K}_3$

$\text{proj}_{\mathbf{K}_3}(\langle 2, 1, 2, 3\rangle) = \left\langle \frac{56}{25}, \frac{11}{25}, \frac{42}{25}, \frac{77}{25}\right\rangle$

(e) $\langle 3, 5, -5, 7\rangle$ onto $\mathbf{K}_3$

$\text{proj}_{\mathbf{K}_3}(\langle 3, 5, -5, 7\rangle) = \left\langle \frac{74}{25}, \frac{382}{75}, -\frac{371}{75}, \frac{524}{75} \right\rangle$

(f) $\langle 3, 5, -5, 7\rangle$ onto $\mathbf{K}_4$

$\text{proj}_{\mathbf{K}_4}(\langle 3, 5, -5, 7\rangle) = \left\langle \frac{16}{3}, \frac{8}{3}, -\frac{8}{3}, 7 \right\rangle$

5. A square matrix $P \in \mathbb{R}^{n\times n}$ whose columns form an orthonormal basis of $\mathbb{R}^n$ is called an *orthogonal matrix*. Prove the following identities. *Hint: Consider the matrix multiplication $PP^T$.*

(a) $P^{-1} = P^T$

Clearly $PP^T = I$ since $PP^T$ is the dot product matrix. Also, $P$ is invertible since its columns form a basis for $\mathbb{R}^n$. Therefore, taking left inverse of both sides yields the result.

(b) $\det(P) = \pm 1$

Note from part (a) that $P\,P^T = I$, with $\det(A^T) = \det(A)$, thus

$$\begin{aligned} \det(P\,P^T) &= \det(I) \\ \det(P)\det(P^T) &= 1 \\ \det(P)\det(P) &= 1 \\ \det(P)^2 &= 1 \\ \det(P) &= \pm 1 \end{aligned}$$

6. Use problem 5 to show that any real matrix $A$ of the form

$$A = \frac{1}{\sqrt{a^2+b^2}} \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

is orthogonal if $a^2 + b^2 \neq 0$.

Under the given assumption, it is easy to verify both property (a) and property (b) of problem 5, and since $P\,P^T = I$, the columns of $A$ form an orthonormal basis for $\mathbb{R}^2$.

7. Show that every $2 \times 2$ rotational matrix, given by

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

for some angle $\theta$, is an orthogonal matrix.

Clearly the rotation matrix given above has the same form as that of $A$ from problem 6. Furthermore, notice that each entry in $A$ satisfies $-1 \leq A_{i,j} \leq 1$. Also, notice that since $\cos(\theta) = \dfrac{a}{\sqrt{a^2+b^2}}$ and $\sin(\theta) = \dfrac{b}{\sqrt{a^2+b^2}}$, we still satisfy $\cos^2(\theta) + \sin^2(\theta) = 1$.

8. Use problem 5 to show that if both $P$ and $Q$ are orthogonal matrices of the same size, then their two products $PQ$ and $QP$ are also orthogonal.

Since $P$ and $Q$ are orthogonal matrices, we have that $PP^T = QQ^T = I$. We need to show that $(PQ)(PQ)^T = I$. However, using the rules of transpose, we get that

$$\begin{aligned}
(PQ)(PQ)^T &= (PQ)(Q^T P^T) \\
&= P(QQ^T)P^T \\
&= P(I)P^T \\
&= PP^T \\
&= I
\end{aligned}$$

which proves that $PQ$ is orthogonal. The same argument hold for $QP$ by swapping $P$ and $Q$ in the above argument.

9. Let $\mathbf{B} = \{\overrightarrow{w_1}, \overrightarrow{w_2}, \ldots, \overrightarrow{w_n}\}$ be any orthogonal basis of $\mathbb{R}^n$. Let $\mathbb{S}$ be the subspace of $\mathbb{R}^n$ with basis $\mathbf{B}_1 = \{\overrightarrow{w_1}, \overrightarrow{w_2}, \ldots, \overrightarrow{w_k}\}$ for $k < n$. What is a basis of $\mathbb{S}^\perp$?

Since $\mathbf{B}$ is orthonormal, then by definition, $\overrightarrow{w_i} \cdot \overrightarrow{w_j} = 0$ for $1 \leq i \leq k$ and $k+1 \leq j \leq n$. Thus setting

$$\mathbf{B}_2 = \{\overrightarrow{w_{k+1}}, \overrightarrow{w_{k+2}}, \ldots, \overrightarrow{w_n}\}$$

will yield a basis for $\mathbb{S}^\perp$.

10. Let $\mathbf{B}$ be any finite orthonormal subset of $\mathbb{R}^n$. Prove that $\mathbf{B}$ is an independent set. Is this also true if $\mathbf{B}$ is merely orthogonal? Is any $n$-element orthonormal subset of $\mathbb{R}^n$ automatically a basis of $\mathbb{R}^n$?

Let us assume, for a second, that there is a vector, $\overrightarrow{v_{k+1}}$ in $\mathbf{B} = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_k}, \overrightarrow{v_{k+1}}\}$ which is linearly dependent. As a result, we can express this vector as follows:

$$\overrightarrow{v_{k+1}} = \sum_{i=1}^{k} a_i \overrightarrow{v_i},$$

where not all of the $a_i$'s are all zero. But since **B** is an orthonormal subset, $\overrightarrow{v_{k+1}} \cdot \overrightarrow{v_j} = 0$ for all $1 \leq j \leq k$. Let us assume that $a_p \neq 0$ (at least one of the $a_i$'s is not zero), then

$$\begin{aligned}
\overrightarrow{v_{k+1}} \cdot \overrightarrow{v_p} &= \sum_{i=1}^{k} a_i \overrightarrow{v_i} \cdot \overrightarrow{v_p} \\
&= a_p \overrightarrow{v_p} \cdot \overrightarrow{v_p} \\
&= a_p \left|\overrightarrow{v_p}\right|^2 > 0
\end{aligned}$$

But this is a contradiction to the fact that all $k+1$ vectors were orthogonal. Note that there was no requirement on the vectors being of unit length, thus it is true if **B** is merely orthogonal. The final question can also be answered in the affirmative.

# Chapter 9

# Linear Maps from $\mathbb{R}^n$ to $\mathbb{R}^m$

## 9.1 Basics About Linear Maps

1. Given a linear map $T : \mathbb{R}^m \to \mathbb{R}^m$, expressed in terms of matrix multiplication by $A\overrightarrow{x} = \overrightarrow{y}$, where $A$ is invertible, does $A^{-1}y = x$ correspond to the inverse map $T^{-1} : \mathbb{R}^m \to \mathbb{R}^m$?

If $T(\overrightarrow{x}) = \overrightarrow{y}$, then we should have that $T^{-1}(\overrightarrow{y}) = \overrightarrow{x}$. If $T$ is represented by the matrix $A$, then we have $A\overrightarrow{x} = \overrightarrow{y}$. Since $A$ is invertible, we can left multiply by $A^{-1}$ on both sides, yielding $A^{-1}A\overrightarrow{x} = A^{-1}\overrightarrow{y}$, or $A^{-1}\overrightarrow{y} = \overrightarrow{x}$, thus $A^{-1}$ is the matrix corresponding to $T^{-1}$.

2. For each of the given linear maps, determine the matrix $A$ such that $T(\overrightarrow{x}) = A\overrightarrow{x}$.

(a) $A\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \to A = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$

(b) $A\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \to A = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$

(c) $A\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \to A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix}$

(d) $A\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \to A = \begin{bmatrix} 2 & -1 & -1 \\ 3 & -1 & -1 \\ 6 & -3 & -1 \end{bmatrix}$

3. Compute the inverse map, $T^{-1}$, to the maps from problem 2 parts (a) and (d).

(a) $A^{-1} = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$ (d) $A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -\frac{3}{2} & 2 & -\frac{1}{2} \\ -\frac{3}{2} & 0 & \frac{1}{2} \end{bmatrix}$

4. So far, we have considered linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ where we know the image of a basis of $\mathbb{R}^n$. Consider the following map:

$$T(\langle 1, 2, 0, 1\rangle) = \langle -5, -1\rangle,$$
$$T(\langle -1, 0, 3, 2\rangle) = \langle 8, 14\rangle$$

Explain why this map cannot be defined by a unique matrix $A \in \mathbb{R}^{2\times 4}$?

We would need to know the image of four independent vectors in $\mathbb{R}^4$ to actually construct a unique mapping (since inverses are unique). With a non-invertible matrix, which is the problem here when we attempt to solve equation (9.14), we are not guaranteed a unique solution, so we either have no solution or an infinite number of solutions.

5. Construct two $2 \times 4$ matrices that satisfy the map given in problem 4.

The following are two matrices which will work:

$$A_1 = \begin{bmatrix} 0 & -3 & 2 & 1 \\ -6 & 2 & 2 & 1 \end{bmatrix}, \ A_2 = \begin{bmatrix} -17 & 6 & -3 & 0 \\ -5 & 2 & 3 & 0 \end{bmatrix}$$

To compute a matrix $A$ which works for this problem, we simply solve the following equation:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 0 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 8 \\ -1 & 14 \end{bmatrix}$$

Notice that this will result in four equations in eight unknowns. So there could possibly be a four-dimensional solution set to this problem.

6. Define $A_b$ and $A_c$ to be the matrices corresponding to the linear maps from parts (b) and (c) of problem 2. Verify the following:

Both of these are straight forward calculations. First, we compute $A_c A_b$:

$$A_c A_b = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 2 & -1 \end{bmatrix}$$

Thus, the composition of mappings applied to each vector are given by:

(a) $A_c A_b \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

(b) $A_c A_b \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

7. Construct a linear map from $\mathbb{R}^2$ to the subspace of $\mathbb{R}^3$ given by

$$\mathbb{S} = \{\langle x, 0, z\rangle \,|\, x, z, \in \mathbb{R}\}$$

The idea here, is that we need to construct a matrix $A \in \mathbb{R}^{3\times 2}$ such that the image under the map $A$ gives $\mathbb{S}$. We can pick a basis of $\mathbb{R}^2$ to send to $\mathbb{S}$, and basis will do, but let us pick the standard basis. This yields the following two equations

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

These equations are easy to solve, as we end up with

$$\begin{bmatrix} a_{1,1} \\ a_{2,1} \\ a_{3,1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} a_{1,2} \\ a_{2,2} \\ a_{3,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus a matrix $A$ which satisfies the desired property is given by

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

8. For the linear map $T$ in Example 9.1.4, find the image $T(\mathbb{S})$ of the subspace

$$\mathbb{S} = \{\, a\,\langle 1, 2, -3\rangle + b\,\langle -4, 5, 1\rangle \mid a, b \in \mathbb{R}\}$$

and inverse image $T^{-1}(\mathbb{K})$ of the subspace

$$\mathbb{K} = \{\, a\,\langle 5, 4, 6, 5\rangle + b\,\langle 6, 5, 5, 9\rangle + c\,\langle -1, 2, -3, 7\rangle \mid a, b, c \in \mathbb{R}\}$$

Give a basis and the dimension for both the image $T(\mathbb{S})$ and the inverse image $T^{-1}(\mathbb{K})$.

The image of $T(\mathbb{S})$ is found by taking linear combinations of $T(\langle 1, 2, -3\rangle)$ and $T(\langle -4, 5, 1\rangle)$, so

$$\begin{aligned} T(\mathbb{S}) &= \{\, T(a\,\langle 1, 2, -3\rangle + b\,\langle -4, 5, 1\rangle) \mid a, b \in \mathbb{R}\} \\ &= \{\, a\,T(\,\langle 1, 2, -3\rangle) + b\,T(\langle -4, 5, 1\rangle) \mid a, b \in \mathbb{R}\} \\ &= \{\, a\,T(\,\langle 1, 2, -3\rangle) + b\,T(\langle -4, 5, 1\rangle) \mid a, b \in \mathbb{R}\} \\ &= \{\, a\,\langle 0, 2, 10, -10\rangle + b\,\langle -24, 7, -29, 9\rangle \mid a, b \in \mathbb{R}\} \end{aligned}$$

Since the two given vectors in the above line are clearly independent, the dimension of $T(\mathbb{S})$ is two.

To find the inverse image of $\mathbb{K}$, we need to find the subspace, $\mathbb{S}$ of $\mathbb{R}^3$ such that $T(\mathbb{S}) \subseteq \mathbb{K} \subset \mathbb{R}^4$. In matrix multiplication form, this looks like

$$\begin{bmatrix} 5 & -1 & 1 \\ 1 & 2 & 1 \\ 7 & 0 & -1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = a \begin{bmatrix} 5 \\ 4 \\ 6 \\ 5 \end{bmatrix} + b \begin{bmatrix} 6 \\ 5 \\ 5 \\ 9 \end{bmatrix} + c \begin{bmatrix} -1 \\ 2 \\ -3 \\ 7 \end{bmatrix}$$

In augmented matrix form, we row reduce to get

$$\text{rref}\left( \begin{bmatrix} 5 & -1 & 1 & 5 & 6 & -1 \\ 1 & 2 & 1 & 4 & 5 & 2 \\ 7 & 0 & -1 & 6 & 5 & -3 \\ 0 & 1 & 4 & 5 & 9 & 7 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We therefore end up with $x = a + b$, $y = a + b$, $z = a + 2b$ and $c = 0$. So the subspace of $\mathbb{R}^3$ that is our solution is

$$\mathbb{S} = \{ a \langle 1, 1, 1 \rangle + b \langle 1, 1, 2 \rangle \mid a, b \in \mathbb{R} \}$$

9. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ have the rule

$$T(\langle x, y \rangle) = x \ \langle \cos(\theta), -\sin(\theta) \rangle + y \ \langle \sin(\theta), \cos(\theta) \rangle$$

Show that $T$ is a linear map and explain what it does geometrically.

Notice that this map can be written as

$$T\left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

which implies that $T$ is simply a rotation transformation. Since we have expressed it as a matrix multiplication, clearly $T$ is a linear transformation.

10. Let $\mathbb{S}$ be a subspace of $\mathbb{R}^n$, and $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $T(\overrightarrow{v}) = \text{proj}_{\mathbb{S}}(\overrightarrow{v})$, for $\overrightarrow{v} \in \mathbb{R}$.

(a) Is the function $T$ a linear map? Explain why if it is, but if it is not give an example to illustrate why not.

Since the projection of a vector can be written as matrix multiplication (see the previous chapter), it is indeed linear.

(b) If the function $T$ in part (a) is a linear map, what is the image $T(\mathbb{R}^n)$?

The image of $T(\mathbb{R}^n)$ is $\mathbb{S}$ since $\mathbb{S} \subseteq \mathbb{R}^n$ and $T(\mathbb{S}) = \mathbb{S}$.

(c) If the function $T$ in part (a) is a linear map, what is $T^{-1}\big(\overrightarrow{0}_n\big)$?

So we must determine which vectors, when projected, end up being $\overrightarrow{0}$. Clearly, if $\overrightarrow{v}$ is a vector orthogonal to a basis of $\mathbb{S}$, then this will be true, as the dot product of $\overrightarrow{v}$ with any vector in a basis for $\mathbb{S}$ must be zero. Therefore, we know that at least $S^{\perp} \subseteq T^{-1}\big(\overrightarrow{0}_n\big)$. Are there any more vectors that could be in $T^{-1}\big(\overrightarrow{0}_n\big)$? The answer is no, due to the reasoning already given.

11. (a) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^k$ be two linear maps. Show that their *composite*, $S \circ T : \mathbb{R}^n \to \mathbb{R}^k$, is also a linear map. Recall that the composite function $S \circ T : \mathbb{R}^n \to \mathbb{R}^k$ has the rule $(S \circ T)(\overrightarrow{v}) = S(T(\overrightarrow{v}))$, for all $\overrightarrow{v} \in \mathbb{R}^n$.

If $T$ is a linear map, then $T(\overrightarrow{x})$ can be represented by a matrix multiplication $A\overrightarrow{x}$, where $A \in \mathbb{R}^{m\times n}$. Similarly, $S$ can be represented by a matrix $B \in \mathbb{R}^{k\times m}$. So $S \circ T$ should be represented by the matrix $BA \in \mathbb{R}^{k\times n}$. To see this, notice that

$$\begin{aligned} S(T(\overrightarrow{v})) &= S(A\overrightarrow{v}) \\ &= BA\overrightarrow{v} \end{aligned}$$

Now since $BA \in \mathbb{R}^{k\times n}$, we automatically get that $S \circ T$ is linear, for

$$\begin{aligned} S \circ T(a\,\overrightarrow{u} + b\,\overrightarrow{v}) &= BA(a\,\overrightarrow{u} + b\,\overrightarrow{v}) \\ &= a\,BA(\overrightarrow{u}) + b\,BA(b\,\overrightarrow{v}) \\ &= a\,S \circ T(\overrightarrow{u}) + b\,S \circ T(\overrightarrow{v}) \end{aligned}$$

(b) Let $\mathbb{K}$ be a subspace of $\mathbb{R}^k$. Explain why the inverse image is

$$(S \circ T)^{-1}(\mathbb{K}) = T^{-1}\left(S^{-1}(\mathbb{K})\right).$$

This is true due to function composition, where $(f \circ g)^{-1} = g^{-1}f^{-1}$. Furthermore, if $A$ and $B$ were square, we could also use the property that $(AB)^{-1} = B^{-1}A^{-1}$.

12. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map that has an inverse function $T^{-1} : \mathbb{R}^m \to \mathbb{R}^n$. Prove that $T^{-1}$ is also a linear map and $n = m$.

For any two vectors in $\overrightarrow{u}$ and $\overrightarrow{v}$, we need to show that $T^{-1}(\overrightarrow{u} + \overrightarrow{v}) = T^{-1}(\overrightarrow{u}) + T^{-1}(\overrightarrow{v})$, and also that $T^{-1}(c\overrightarrow{u}) = cT^{-1}(\overrightarrow{u})$ for an arbitrary scalar *c.* First note that since $\overrightarrow{u}$ and $\overrightarrow{v}$ are in the image of $T$, there exists $\overrightarrow{x}$ and $\overrightarrow{y}$ such that $T(\overrightarrow{x}) = \overrightarrow{u}$ and $T(\overrightarrow{y}) = \overrightarrow{v}$. Therefore,

$$\begin{aligned} T^{-1}(\overrightarrow{u} + \overrightarrow{v}) &= T^{-1}\left(T(\overrightarrow{x}) + T(\overrightarrow{y})\right) \\ &= T^{-1}\left(T(\overrightarrow{x} + \overrightarrow{y})\right) \\ &= \overrightarrow{x} + \overrightarrow{y} \\ &= T^{-1}(\overrightarrow{u}) + T^{-1}(\overrightarrow{v}) \end{aligned}$$

Similarly,

$$\begin{aligned} T^{-1}(c\,\overrightarrow{u}) &= T^{-1}\left(c\,T(\overrightarrow{x})\right)) \\ &= T^{-1}\left(T(c\,\overrightarrow{x})\right)) \\ &= \overrightarrow{c\,x} \\ &= c\,T^{-1}(\overrightarrow{u}) \end{aligned}$$

13. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map which has an inverse linear map $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$. Prove that if $T$ can be written as the matrix multiplication $T(\overrightarrow{x}) = A\overrightarrow{x}$, then $T^{-1}$ can be written as the matrix multiplication $T^{-1}(\overrightarrow{y}) = A^{-1}\overrightarrow{y}$.

Let us assume that the matrix corresponding to $T^{-1}$ is $B$. From problem 11, we know that $T^{-1}(T(\overrightarrow{x})) = \overrightarrow{x}$, and thus in matrix form, we have $BA\overrightarrow{x} = \overrightarrow{x}$, thus $BA = I_n$. Similarly, we also have that $AB = I_n$ by considering $T\left(T^{-1}(\overrightarrow{x})\right) = \overrightarrow{x}$. Notice that this is the definition of $B$ being the multiplicative inverse to $A$, and thus $T^{-1}$ has matrix representation $A^{-1}$.

14. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map and $\mathbf{B}$ be a basis of $\mathbb{R}^n$. Show that $T$ has an inverse linear map $T^{-1}$ if and only if $T(\mathbf{B})$ is also a basis of $\mathbb{R}^n$. This says that $T$ is invertible if and only if $T$ sends a basis to a basis.

If $T$ has matrix representation $A$, then consider the matrix $W$ whose columns are the basis vectors chosen for $\mathbb{R}^n$. Let $V$ be the matrix whose columns are the basis vectors $T(\mathbf{B})$. This gives us the matrix equation $AW = V$. Note that $W$ and $V$ are both square invertible matrices since their columns are linearly independent. Therefore, we have $A = VW^{-1}$, and $A^{-1} = WV^{-1}$.

Now to prove the argument, it is obvious that if $T(\mathbf{B})$ is also a basis of $\mathbb{R}^n$, then $T^{-1}$ exists since $V$ is invertible and we can solve for $A^{-1}$, which is the matrix representation of $T^{-1}$ by the previous problem.

If $T$ does have an inverse linear map, then $T(\mathbf{B})$ must be a basis for $\mathbb{R}^n$ because one can compute the inverse matrix to $V$, since both $A$ and $W$ are invertible. Therefore the columns of $V$ are linearly independent, and thus form a basis for $\mathbb{R}^n$, but $V$ is $T(\mathbf{B})$.

15. How can you define a linear map $T : \mathbb{V} \to \mathbb{W}$, for $\mathbb{V}$ a subspace of $\mathbb{R}^n$ and $\mathbb{W}$ a subspace of $\mathbb{R}^m$? Explain how the material of this section can be altered for this new more general linear map. Does everything about linear maps also work if we replace $\mathbb{R}$ by $\mathbb{C}$ and even mix $\mathbb{R}$ and $\mathbb{C}$.

The problems arise when attempting to compute inverses of subspaces, as the matrix representations of such transformations will not be square. The transformations can be done by restricting domains (thus almost like considering images of subspaces etc...). Of course, the transformations will be invertible if you restrict the range to be the image of the subspace used as the domain but you cannot find the inverse by simply taking a matrix inverse. These concepts also generalize to $\mathbb{C}$.

## 9.2 The Kernel and Image Subspaces of a Linear Map

1. Each of the following matrices represent a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$. Compute both the $\text{Im}(T)$ and $\text{Ker}(T)$ for each map, expressing your answer in terms of basis vectors.

(a) $\begin{bmatrix} 2 & -3 \\ 8 & -4 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & -8 & -2 \\ -4 & 16 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & -8 & -2 \\ -4 & 16 & 1 \\ -2 & 8 & -1 \end{bmatrix}$ (e) $\begin{bmatrix} 5 & -2 & -1 \\ -4 & 2 & 1 \\ -2 & 1 & -4 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 3 & 0 & -2 \\ 0 & 1 & 3 & 3 \end{bmatrix}$

2. For each of the maps from problem 1, verify that Theorem **??** holds.

3. Classify each map from problem 1 as one-to-one, onto or bijective; if the map does not satisfy any of the given properties, then state so.

4. Compute both the $\text{Im}(T)$ and $\text{Ker}(T)$ for each of the following maps.

(a) $T(\langle x, y\rangle) = \langle x, -x, x\rangle$

(b) $T(\langle x, y\rangle) = \langle y, x, y, x\rangle$

(c) $T(\langle x, y, z\rangle) = \langle x + y, y - z, x - y\rangle$

(d) $T(\langle x, y, z\rangle) = x + y + z$

5. Classify each map from problem 4 as one-to-one, onto or bijective; if the map does not satisfy any of the given properties, then state so.

6. In problems 1 and 4, find a basis of $\text{Ker}(T)^{\perp}$ and show that:

$$\dim\left(\text{Ker}(T)^{\perp}\right) = \dim(\text{Im}(T))$$

7. Prove that a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is not one-to-one if $m < n$.

8. Prove that a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is not onto if $m > n$.

9. (See *Homework* problem 11 of Section 9.1). Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^k$ be two linear maps. Their composite $S \circ T : \mathbb{R}^n \to \mathbb{R}^k$ is also a linear map.

(a) Let $S \circ T$ be one-to-one. Then must both $S$ and $T$ be one-to-one? If yes, explain why. If no, then give an example of why not.

(b) Let $S \circ T$ be onto. Then must both $S$ and $T$ be onto? If yes, explain why. If no, then give an example of why not.

10. Let $A$ be the $n \times n$ matrix representing the linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ through multiplication by $A$.

(a) Explain how the columns of the matrix $A$ are found and what they are in the range $\mathbb{R}^n$.

(b) Explain why $\det(A) \neq 0$ if and only if $T$ sends a basis to a basis.

## 9.3 Composites of Two Linear Maps and Inverses

1. For each of the following pairs of maps, determine which order, if possible, $S$ and $T$ can be composed in.

| | | | | | |
|---|---|---|---|---|---|
| (a) | $T : \mathbb{R}^3 \to \mathbb{R}^5$<br>$S : \mathbb{R}^3 \to \mathbb{R}^3$ | (b) | $T : \mathbb{R}^3 \to \mathbb{R}^5$<br>$S : \mathbb{R}^5 \to \mathbb{R}^3$ | (c) | $T : \mathbb{R}^2 \to \mathbb{R}$<br>$S : \mathbb{R} \to \mathbb{R}^3$ |
| (d) | $T : \mathbb{R}^5 \to \mathbb{R}^3$<br>$S : \mathbb{R}^4 \to \mathbb{R}^3$ | (e) | $T : \mathbb{R} \to \mathbb{R}^2$<br>$S : \mathbb{R}^4 \to \mathbb{R}$ | (f) | $T : \mathbb{R}^2 \to \mathbb{R}^3$<br>$S : \mathbb{R}^3 \to \mathbb{R}^3$ |

2. For each of the following pairs of maps, compute $S \circ T$ without using matrices.

(a) $T(\langle x, y \rangle) = x + y$
$S(a) = \langle a, -a \rangle$

(b) $T(x) = \langle x, -x, 2x \rangle$
$S(\langle a, b, c \rangle) = \langle a + b, a - c \rangle$

(c) $T(\langle x, y \rangle) = \langle -x, y, x + y \rangle$
$S(\langle a, b, c \rangle) = \langle -a, b \rangle$

(d) $T(x) = \langle 3x, 2x, -4x \rangle$
$S(\langle a, b, c \rangle) = a + b + c$

3. The compositions from problem 2 should have yielded two such that $(S \circ T)(\overrightarrow{v}) = \overrightarrow{v}$, for all $\overrightarrow{v}$ in the domain of $T$. Which two are they?

4. The two compositions found in problem 3 highlight an important sticking point in Definition **??** and Theorem **??**. Verify that $(T \circ S)(\overrightarrow{v}) \neq \overrightarrow{v}$, for all $\overrightarrow{v}$ in the domain of $S$. Why is this the case?

5. Determine the linear map given by each of the following matrices.

(a) $\begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (f) $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

6. Four of the matrices from problem 5 correspond to linear maps, $S$ or $T$, from problem 2. Find them and state which maps they correspond to.

7. If the matrix $C$, which represents a composite of two linear maps through multiplication by $C$, has nonzero determinant, then must the same be true for each of the two matrices $A$ and $B$, which represent the individual linear maps? If yes, then explain why. If not, then give an example to verify it.

8. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map that is invertible as a function. Show directly that its inverse function, $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$, is also a linear map.

## 9.4 Change of Bases for the Matrix Representation of a Linear Map

1. Represent each vector of the standard basis $\mathbf{S}_2$ of $\mathbb{R}^2$ as a linear combination of vectors in the basis $\mathbf{B} = \{\langle 1, 1 \rangle, \langle 1, -1 \rangle\}$. Also, write your answer in vector

form, using the correct notation.

2. Express each of the following vectors in $\mathbb{R}^2$ in vector form using the basis $\mathbf{B}$ from problem 1:

| | | | | | |
|---|---|---|---|---|---|
| (a) | $\langle 2,0\rangle$ | (b) | $\langle 2,-3\rangle$ | (c) | $\langle 9,-1\rangle$ |
| (d) | $\langle -2,0\rangle$ | (e) | $\langle -\frac{1}{2},\frac{3}{2}\rangle$ | (f) | $\langle 7,-6\rangle$ |

3. Verify that the following equation has only the trivial solution:

$$a\overrightarrow{e_1}+b\overrightarrow{e_2}=a\langle 1,1\rangle+b\langle 1,-1\rangle$$

Here, $\overrightarrow{e_1}$ and $\overrightarrow{e_2}$ are the standard basis vectors for $\mathbb{R}^2$. What does this imply in regards to representation of vectors with the standard basis and the basis $\mathbf{B}$ from problem 1?

4. Represent each vector of the standard basis $\mathbf{S}_3$ of $\mathbb{R}^3$ as a linear combination of vectors in the basis $\mathbf{B}=\{\langle 1,0,1\rangle,\langle 0,1,1\rangle,\langle 0,-1,0\rangle\}$. Also, write your answer in vector form, using the correct notation.

5. Express each of the following vectors in $\mathbb{R}^3$ in vector form using the basis $\mathbf{B}$ from problem 4.

| | | | | | |
|---|---|---|---|---|---|
| (a) | $\langle 1,1,0\rangle$ | (b) | $\langle 2,2,-2\rangle$ | (c) | $\langle -5,6,6\rangle$ |
| (d) | $\langle 2,-2,3\rangle$ | (e) | $\langle 1,1,1\rangle$ | (f) | $\langle 3,-4,-10\rangle$ |

6. Construct a matrix $A'$ that takes vectors in $\mathbb{R}^2$ expressed in terms of the basis $\mathbf{B}_1=\{\langle 1,1\rangle,\langle 1,-1,\rangle\}$, and expresses them in terms of the basis $\mathbf{B}_2=\{\langle 2,-1\rangle,\langle 3,2,\rangle\}$, i.e., $A'\overrightarrow{x}_{\mathbf{B}_1}=\overrightarrow{x}_{\mathbf{B}_2}$.

7. Construct a matrix $A'$ that takes vectors in $\mathbb{R}^3$ expressed in terms of the basis $\mathbf{B}_1=\{\langle 1,0,1\rangle,\langle 0,1,1\rangle,\langle 0,-1,0\rangle\}$, and expresses them in terms of the basis $\mathbf{B}_2=\{\langle 1,1,-1\rangle,\langle 1,-1,1\rangle,\langle 1,-1,-1\rangle\}$.

8. Given the map $T:\mathbb{R}^2\to\mathbb{R}^3$ defined by $T(\langle x,y\rangle)=\langle 0,y,x\rangle$, find the matrix $A'$ corresponding to $T$ under the two bases $\mathbf{B}_2=\{\langle 1,1\rangle,\langle 1,-1\rangle\}$ and $\mathbf{B}_3=\{\langle 0,1,1\rangle,\langle 1,0,0\rangle,\langle 0,1,0\rangle\}$.

9. Construct the commutation diagram for the map from problem 8.

10. Given the map $S:\mathbb{R}^3\to\mathbb{R}^4$ defined by $S(\langle x,y,z\rangle)=\langle 0,z,y,x\rangle$, find the

matrix $A'$ corresponding to $S$ under the two bases

$$\mathbf{B}_3 = \{\langle 0,1,1\rangle, \langle 1,0,0\rangle, \langle 0,1,0\rangle\}$$

and

$$\mathbf{B}_4 = \{\langle 0,1,1,1\rangle, \langle 1,0,0,1\rangle, \langle 0,1,1,0\rangle, \langle 0,0,1,1\rangle\}$$

11. Construct the commutation diagram for the map from problem 10.

12. Construct the commutation diagram for $S \circ T$, where $T$ and $S$ are the maps from problems 8 and 10, respectively. Use the diagram to find the matrix corresponding to the map $S \circ T$.

13. Consider the case of a linear map whose domain is represented by a non-standard basis $\mathbf{B_n}$, and whose image is also represented by a non-standard basis $\mathbf{B}_m$. Hence, we already have $T'\left(\overrightarrow{x}_{\mathbf{B}_n}\right) = \overrightarrow{y}_{\mathbf{B}_m}$. How can you recover the original maps's matrix $A$ in the standard bases, given the matrix $A'$ that represents $T'$ for the pair of nonstandard bases? *Hint: Drawing a commutation diagram can help.*

14. In problems 6 and 7, verify directly that $(A')^{-1}$ reverses the order of the two bases.

15. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map where we have two (different) pairs of bases, bases $\mathbf{B}$ and $\mathbf{C}$ for $\mathbb{R}^n$ and bases $\mathbf{D}$ and $\mathbf{E}$ for $\mathbb{R}^m$. Let $T_{\mathbf{D}}^{\mathbf{B}}$ be the matrix that represents the linear map $T$ in the two bases $\mathbf{B}$ on the domain $\mathbb{R}^n$ and $\mathbf{D}$ on the range $\mathbb{R}^m$. Similarly, $T_{\mathbf{E}}^{\mathbf{C}}$ be the matrix that represents the linear map $T$ in the two bases $\mathbf{C}$ on the domain $\mathbb{R}^n$ and $\mathbf{E}$ on the range $\mathbb{R}^m$. Also, for $I$ the identity linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$, we have the matrix $I_{\mathbf{C}}^{\mathbf{B}}$ that represents the linear map $I$ in the two bases $\mathbf{B}$ and $\mathbf{C}$ while similarly, the matrix $I_{\mathbf{D}}^{\mathbf{E}}$ represents the identity linear map $I$ from $\mathbb{R}^m$ to $\mathbb{R}^m$ in the two bases $\mathbf{D}$ and $\mathbf{E}$.

(a) Explain the meaning of, and discuss the validity of, the following commutative diagram:

$$\begin{array}{ccc}
\mathbb{R}^n_{\mathbf{B}} & \xrightarrow{T_{\mathbf{D}}^{\mathbf{B}}} & \mathbb{R}^m_{\mathbf{D}} \\
I_{\mathbf{C}}^{\mathbf{B}} \big\downarrow & & \big\uparrow I_{\mathbf{D}}^{\mathbf{E}} \\
\mathbb{R}^n_{\mathbf{C}} & \xrightarrow{T_{\mathbf{E}}^{\mathbf{C}}} & \mathbb{R}^m_{\mathbf{E}}
\end{array}$$

(b) Is the matrix equation $T_{\mathbf{D}}^{\mathbf{B}} = I_{\mathbf{D}}^{\mathbf{E}} T_{\mathbf{E}}^{\mathbf{C}} I_{\mathbf{C}}^{\mathbf{B}}$ correct? Explain your reasoning.

(c) Verify with an example the matrix equation in part (b) when only one pair $\mathbf{B}$ and $\mathbf{D}$ of bases are the standard bases of $\mathbb{R}^2$.

(d) Verify with an example the matrix equation in part (b) when all four bases are not the standard bases of $\mathbb{R}^2$.

16. (Continuation of problem 15.) Let $I$ be the identity linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ and $\mathbf{B}$, $\mathbf{C}$ be two bases of $\mathbb{R}^n$.

(a) Explain why $I_{\mathbf{B}}^{\mathbf{B}} = I_{\mathbf{C}}^{\mathbf{C}} = I_n$, where $I_n$ is the $n \times n$ identity matrix.

(b) Explain why $I_{\mathbf{C}}^{\mathbf{B}} = \left(I_{\mathbf{B}}^{\mathbf{C}}\right)^{-1}$.

(c) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Explain why

$$T_{\mathbf{B}}^{\mathbf{B}} = I_{\mathbf{B}}^{\mathbf{C}} T_{\mathbf{C}}^{\mathbf{C}} I_{\mathbf{C}}^{\mathbf{B}}$$

is correct, and thus

$$T_{\mathbf{B}}^{\mathbf{B}} = I_{\mathbf{B}}^{\mathbf{C}} T_{\mathbf{C}}^{\mathbf{C}} \left(I_{\mathbf{B}}^{\mathbf{C}}\right)^{-1}$$

This last equation says that the two $n \times n$ matrices $T_{\mathbf{B}}^{\mathbf{B}}$ and $T_{\mathbf{C}}^{\mathbf{C}}$ are *similar* matrices.

(d) Explain why $\left(T^{-1}\right)_{\mathbf{C}}^{\mathbf{B}} = \left(T_{\mathbf{B}}^{\mathbf{C}}\right)^{-1}$.

(e) Why is $\det\left(T_{\mathbf{C}}^{\mathbf{C}}\right) = \det\left(T_{\mathbf{B}}^{\mathbf{B}}\right)$?

(f) Let $n = 2$, and do examples to illustrate parts (a)–(e) above.